Screening of cosmological constant for De Sitter Universe in non-local gravity, phantom-divide crossing and finite-time future singularities

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Abstract

We investigate de Sitter solutions in non-local gravity as well as in non-local gravity with Lagrange constraint multiplier. We examine a condition to avoid a ghost and discuss a screening scenario for a cosmological constant in de Sitter solutions. Furthermore, we explicitly demonstrate that three types of the finite-time future singularities can occur in non-local gravity and explore their properties. In addition, we evaluate the effective equation of state for the universe and show that the late-time accelerating universe may be effectively the quintessence, cosmological constant or phantom-like phases. In particular, it is found that there is a case in which a crossing of the phantom divide from the non-phantom (quintessence) phase to the phantom one can be realized when a finite-time future singularity occurs. Moreover, it is demonstrated that the addition of an R^2 term can cure the finite-time future singularities in non-local gravity. It is also suggested that in the framework of non-local gravity, adding an R^2 term leads to possible unification of the early-time inflation with the late-time cosmic acceleration.

I. INTRODUCTION

It has been suggested that the current expansion of the universe is accelerating by recent cosmological observations such as Supernovae Ia (SNe Ia) [1], cosmic microwave background (CMB) radiation [2, 3], large scale structure (LSS) [4], baryon acoustic oscillations (BAO) [5], and weak lensing [6]. There are two representative approaches to account for the late time cosmic acceleration. One is the introduction of cosmological constant dark energy in the framework of general relativity (for a review, see, e.g., [7]). The other is the modification of gravity, for example, $F(R)$ gravity, where $F(R)$ is an arbitrary function of the scalar curvature R (for recent reviews on $F(R)$ gravity, see, e.g., [8, 9]).

As a possible modification of gravity, non-local gravity produced by quantum effects has been proposed in Ref. [10]. A modification of non-local gravity with an $F(R)$ term has been studied in order to realize a unified scenario of the early-time inflation with the late-time cosmic acceleration [11]. Moreover, there was a proposal on the solution of the cosmological constant problem by non-local modification of gravity [12]. Recently, an explicit mechanism to screen a cosmological constant in non-local gravity has been discussed in Ref. [13]. It is also significant to mention that there exists the problem of ghosts in non-local gravity [13]. Non-local gravity has been extensively examined in the literature [14–17].

Furthermore, it is known that so-called matter instability [18] occurs in $F(R)$ gravity. This implies that the curvature inside matter sphere becomes very large and hence the curvature singularity could appear. The problem of singularity in the relativistic star formation process was discussed in Ref. [19]. In addition, the instability in $F(R)$ gravity has recently been studied for a gravitating system with a time dependent mass density such as astronomical massive objects in Ref. [20]. The generation mechanism of the time-dependent matter instability in the star collapse has also been investigated in Ref. [21]. Note that a scenario to cure the star singularity via adding an R^2 term or some other powers of the scalar curvature has been developed in Refs. [8, 21]. On the other hand, the stability of cosmological perturbations in non-local gravity has been explored in Ref. [16]. It is important to examine whether there exists the curvature singularity in non-local gravity. In what follows, the curvature singularity is called "the finite-time future singularities" throughout this paper.

In this paper, we explore de Sitter solutions not only in non-local gravity but also in non-local gravity with Lagrange constraint multiplier. We also study a condition to avoid a

ghost and present a screening scenario for a cosmological constant in de Sitter solutions. We reconfirm that the problem of ghosts persists in these non-local models. This is an important result of this work. In addition, we explicitly show that three types of the finite-time future singularities can occur in non-local gravity and examine their properties. Furthermore, we investigate the behavior of the effective equation of state for the universe when the finite-time future singularities occur and show that the late-time accelerating universe may be effectively the quintessence, cosmological constant or phantom-like phases. In particular, we demonstrate that there is a case with realizing a crossing of the phantom divide from the non-phantom (quintessence) phase to the phantom one in the limit of the appearance of a finite-time future singularity. Moreover, we consider the cosmological consequences of adding an R^2 term and demonstrate that the addition of an R^2 term can remove the finite-time future singularities in non-local gravity. We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/M_{\mathrm{Pl}}{}^2$ with the Planck mass of $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19} \mathrm{GeV}$.

The paper is organized as follows. In Sec. II, we review non-local gravity and study a de Sitter solution in non-local gravity. We examine a condition to avoid a ghost and discuss a screening scenario for a cosmological constant in the de Sitter solution. In Sec. III, we also investigate a de Sitter solution in non-local gravity with Lagrange constraint multiplier. In Sec. IV, we demonstrate that three types of the finite-time future singularities can occur in non-local gravity and explore their properties. In addition, we consider the behavior of the effective equation of state for the universe when the finite-time future singularities occur in Sec. V. Finally, conclusions are given in Sec. VI.

II.   DE SITTER SOLUTION IN NON-LOCAL GRAVITY

A.   de Sitter solution

In the previous work [13], it has been found that there exists a flat space-time solution even in the presence of the cosmological constant in non-local gravity, and a possible mechanism to screen the cosmological constant has been discussed. The new point in this paper is it is shown that there could exist a de Sitter solution and a effective screening scenario in the de Sitter solution is considered.

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The starting action of non-local gravity is given by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\Box^{-1}R) \right) - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \left(Q; g \right) \right\} . \tag{2.1}$$

Here, g is the determinant of the metric tensor $g_{\mu\nu}$, f is some function, $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ with ∇_μ being the covariant derivative is the covariant d'Almbertian for a scalar field, Λ is a cosmological constant, and $\mathcal{L}_{\text{matter}} \left(Q; g \right)$ is the matter Lagrangian, where Q stands for the matter fields. We regard the action (2.1) as a low energy effective action including quantum corrections. Then in the following, we treat this model only classically. We should also note that the action should not be general action including all the terms allowed by symmetry. This is mainly due to the technical reason since we cannot work with such a complicated action.

We also remark that in the action (2.1), the ultraviolet divergences would become worse, and at large momenta the non-local term will contribute a large correction. We regard, however, that the model could be a low energy effective action including quantum corrections and hence this model should be treated only classically and we do not include further quantum corrections. We do not always consider the relation with string theory, either.

The above action in Eq. (2.1) can be rewritten by introducing two scalar fields η and ξ in the following form:

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\eta) \right) + \xi \left(\Box \eta - R \right) - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \right\} \\ &= \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\eta) \right) - \partial_\mu \xi \partial^\mu \eta - \xi R - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \right\} . \end{aligned} \tag{2.2}$$

We take the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i \right)^2 . \tag{2.3}$$

We consider the case in which the scalar fields η and ξ only depend on time. In this background, the gravitational field equations are given by

$$0 = -3H^2 \left(1 + f(\eta) - \xi \right) + \frac{1}{2} \dot{\xi} \dot{\eta} - 3H \left(f'(\eta) \dot{\eta} - \dot{\xi} \right) + \Lambda + \kappa^2 \rho_{\text{m}} , \tag{2.4}$$

$$0 = \left(2\dot{H} + 3H^2 \right) \left(1 + f(\eta) - \xi \right) + \frac{1}{2} \dot{\xi} \dot{\eta} + \left(\frac{d^2}{dt^2} + 2H \frac{d}{dt} \right) \left(f(\eta) - \xi \right) - \Lambda + \kappa^2 P_{\text{m}} , \tag{2.5}$$

where $H = \dot{a}/a$ is the Hubble parameter, the dot denotes the time derivative of $\partial/\partial t$, and ρ_{m} and P_{m} are the energy density and pressure of matter, respectively. For a perfect fluid of

matter, we have $T_{\text{matter}\,00} = \rho_{\text{m}}$ and $T_{\text{matter}\,ij} = P_{\text{m}}\delta_{ij}$. Furthermore, the equations of motion for the scalar fields η and ξ are given by

$$0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2\,, \tag{2.6}$$

$$0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right)f'(\eta)\,, \tag{2.7}$$

where we have used $R = 6\dot{H} + 12H^2$.

We note that in Ref. [10], the following points are stated. The reason why the non-local contribution $\Box^{-1}$ is taken is that it is a simplest choice for the the inverse of some differential operator to provide the required time lag between the transition from radiation dominance to matter dominance at the radiation-matter equality time $t_{\text{eq}} \sim 10^5$ years. Much larger values can be obtained through other operators, for example, the Paneitz operator arising in the context of conformal anomalies [22], which is given by $(1/\sqrt{-g})\triangle_P$. One gets about 10^6 from the dimensionless combination of the inverse of this operator acting on R^2.

We now assume a de Sitter solution $H = H_0$, where H_0 is a constant. Then, Eq. (2.6) can be solved as

$$\eta = -4H_0 t - \eta_0 e^{-3H_0 t} + \eta_1\,, \tag{2.8}$$

with constants of integration, η_0 and η_1. For simplicity, we only consider the case that $\eta_0 = \eta_1 = 0$. We also suppose $f(\eta)$ is given by

$$f(\eta) = f_0 e^{\frac{\eta}{\beta}} = f_0 e^{-\frac{4H_0 t}{\beta}}\,, \tag{2.9}$$

where f_0 and β are constants. In this case, Eq. (2.7) can be solved as follows,

$$\xi = -\frac{3f_0\beta}{3\beta - 4} e^{-\frac{4H_0 t}{\beta}} + \frac{\xi_0}{3H_0} e^{-3H_0 t} - \xi_1\,, \tag{2.10}$$

where ξ_0 and ξ_1 are constants. For the de Sitter space, a behaves as $a = a_0 e^{H_0 t}$, where a_0 is a constant. Hence, for the matter with the constant equation of state $w_{\text{m}} \equiv P_{\text{m}}/\rho_{\text{m}}$, we find

$$\rho_{\text{m}} = \rho_{\text{m}0} e^{-3(w_{\text{m}}+1)H_0 t}\,, \tag{2.11}$$

where $\rho_{\text{m}0}$ is a constant. By substituting Eqs. (2.8), (2.10) with putting $\xi_0 = 0$, and (2.11) into Eq. (2.4), we obtain

$$0 = -3H_0^2\left(1 + \xi_1\right) + 6H_0^2 f_0 \left(\frac{2}{\beta} - 1\right) e^{-\frac{4H_0 t}{\beta}} + \Lambda + \kappa^2 \rho_{\text{m}0} e^{-3(w_{\text{m}}+1)H_0 t}\,. \tag{2.12}$$

For $\rho_{m0} = 0$, if we choose

$$\beta = 2 , \quad \xi_1 = -1 + \frac{\Lambda}{3H_0^2} , \tag{2.13}$$

de Sitter space can be a solution. Even if $\rho_m \neq 0$, by taking

$$\beta = \frac{4}{3(1 + w_m)} , \quad f_0 = -\frac{\kappa^2 \rho_{m0}}{3H_0^2 (1 + 3w_m)} , \quad \xi_1 = -1 + \frac{\Lambda}{3H_0^2} , \tag{2.14}$$

we find that there is a de Sitter solution.

The second equation in (2.13) or third one in (2.14) implies that

$$H_0^2 = \frac{\Lambda}{3 (1 + \xi_1)} . \tag{2.15}$$

This means that the cosmological constant Λ is effectively screened by ξ. This is one of main results in this paper. We include the cosmological term in the action (2.1) in order to show that the cosmological constant is surely screened by the non-local effect. We should also note that in the case that there does not exist the cosmological constant, i.e., $\Lambda = 0$, if we choose $\xi_1 = -1$, H_0 can be arbitrary. Thus, H_0 can be determined by an initial condition. Since H_0 can be small or large, the theory with the function in Eq. (2.9) with $\beta = 2$ in the first equation in (2.13) could describe the early-time inflation or current cosmic acceleration.

In the presence of matter with $w_m \neq 0$, for $\Lambda = 0$, we may have a de Sitter solution $H = H_0$ even if $f(\eta)$ is given by

$$f(\eta) = f_0 e^{\eta/2} + f_1 e^{3(w_m+1)\eta/4} . \tag{2.16}$$

Therefore, the following solution exists:

$$\eta = -4H_0 t , \quad \xi = 1 - 3f_0 e^{-2H_0 t} + \frac{f_1}{w_m} e^{-3(w_m+1)H_0 t} , \quad \rho_m = -\frac{3(3w_m + 1)H_0^2 f_1}{\kappa^2} e^{-3(1+w_m)H_0 t} . \tag{2.17}$$

Finally we consider the meaning of the constants of the integration η_0 and η_1 in (2.8) and ξ_0 and ξ_1 in (2.10). First we should note that η_1 can be absorbed into the shift of the time coordinate t as $t \to t + \eta_1/4H_0$ then we can always choose $\eta_0 = 0$. Second as clear from (2.15), the constant ξ_1 determine the initial value of scalar curvature $R = 12H_0^2$, in the de Sitter space, where the scalar curvature becomes a constant. We have chosen $\eta_0 = \xi_0 = 0$ for the technical reason to obtain an exact solution. These conditions $\eta_0 = \xi_0 = 0$ give, however, the initial conditions $\eta a^3, \xi a^3 \to 0$ when $t \to -\infty$.

B. Condition to be free of ghost

The existence of a ghost is usually regarded as fatal to the viability of a theory. But it is not necessarily harmful at the classical level [23]. Even at the quantum level, it may depend on how one interprets the degree of freedom associated with a ghost. For example, it has been argued that ghosts may be eliminated in the Euclidean path integral approach [24]. The ghost surely often generates serious problems in the quantum theory like negative norm etc. Even in the classical level, the ghost often generates some problem when we consider the cosmological perturbation etc. since the energy of the ghost is unbounded below. Therefore the absence of a ghost is certainly a desirable property of a theory even if we treat the theory classically. Here we derive the condition to avoid the appearance of a ghost.

To examine the ghost-free condition, we make a conformal transformation to the Einstein frame:

$$g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(\mathrm{E})}, \quad R = \frac{1}{\Omega^2} \left[R^{(\mathrm{E})} - 6 \left(\Box \ln \Omega + g^{(\mathrm{E})\,\mu\nu} \nabla_\mu \ln \Omega \nabla_\nu \ln \Omega \right) \right], \tag{2.18}$$

with

$$\Omega^2 = \frac{1}{1 + f(\eta) - \xi}. \tag{2.19}$$

We use a superscription (E) to represent quantities in the Einstein frame. Then, the action in Eq. (2.2) gives

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left\{ \frac{1}{2\kappa^2} \left[R^{(\mathrm{E})} - 6 \left(\Box \ln \Omega + g^{(\mathrm{E})\,\mu\nu} \nabla_\mu \ln \Omega \nabla_\nu \ln \Omega \right) \right. \right.$$
$$\left. \left. - \Omega^2 g^{\mu\nu} \nabla_\mu \xi \nabla_\nu \eta - 2\Omega^4 \Lambda \right] + \Omega^4 \mathcal{L}_{\mathrm{matter}} \left(Q; \Omega^2 g^{(\mathrm{E})} \right) \right\}. \tag{2.20}$$

The $\Box \ln \Omega$ term may be discarded because it is a total divergence. Hence, we obtain

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left[\frac{1}{2\kappa^2} \left(R^{(\mathrm{E})} - 6 g^{(\mathrm{E})\,\mu\nu} \nabla_\mu \ln \Omega \nabla_\nu \ln \Omega - \Omega^2 g^{\mu\nu} \nabla_\mu \xi \nabla_\nu \eta - 2\Omega^4 \Lambda \right) \right.$$
$$\left. + \Omega^4 \mathcal{L}_{\mathrm{matter}} \left(Q; \Omega^2 g^{(\mathrm{E})} \right) \right]$$
$$= \int d^4x \sqrt{-g^{(\mathrm{E})}} \left[\frac{1}{2\kappa^2} \left(R^{(\mathrm{E})} - 6 g^{(\mathrm{E})\,\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \mathrm{e}^{2\phi} g^{\mu\nu} \nabla_\mu \xi \nabla_\nu \eta - 2\mathrm{e}^{4\phi} \Lambda \right) \right.$$
$$\left. + \mathrm{e}^{4\phi} \mathcal{L}_{\mathrm{matter}} \left(Q; \mathrm{e}^{2\phi} g^{(\mathrm{E})} \right) \right], \tag{2.21}$$

where

$$\phi = \ln \Omega = -\frac{1}{2} \ln \left(1 + f(\eta) - \xi \right). \tag{2.22}$$

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Instead of η and ξ, we may regard ϕ and η to be independent fields. Inserting

$$\xi = -\mathrm{e}^{-2\phi} + (1 + f(\eta)) \tag{2.23}$$

into the action in Eq. (2.21), we finally arrive at

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left\{ \frac{1}{2\kappa^2} \left[R^{(\mathrm{E})} - 6\nabla^\mu\phi\nabla_\mu\phi - 2\nabla^\mu\phi\nabla_\mu\eta - \mathrm{e}^{2\phi}f'(\eta)\nabla^\mu\eta\nabla_\mu\eta - 2\mathrm{e}^{4\phi}\Lambda \right] \right.$$
$$\left. + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q; \mathrm{e}^{2\phi}g^{(\mathrm{E})}\right) \right\}. \tag{2.24}$$

In order to avoid a ghost, the determinant of the kinetic term must be positive, which means [13]

$$\det \begin{vmatrix} 6 & 1 \\ 1 & \mathrm{e}^{2\phi}f'(\eta) \end{vmatrix} = 6\mathrm{e}^{2\phi}f'(\eta) - 1 > 0. \tag{2.25}$$

This condition is assumed to be satisfied. In particular, $f'(\eta) > 0$ is a necessary condition. Thus, the ghost-free condition is expressed as

$$f'(\eta) > \frac{1}{6\mathrm{e}^{2\phi}} = \frac{1 + f(\eta) - \xi}{6} > 0, \tag{2.26}$$

where in the equality of the middle term, we have used Eq. (2.22). Then, we may introduce a new field χ, given by

$$\chi = \int^\eta \sqrt{f'(\eta)}d\eta, \tag{2.27}$$

and rewrite the action in Eq. (2.24) in the form,

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left\{ \frac{1}{2\kappa^2} \left[R^{(\mathrm{E})} - 6\nabla^\mu\phi\nabla_\mu\phi - \frac{2}{\sqrt{f'}}\nabla^\mu\phi\nabla_\mu\chi - \mathrm{e}^{2\phi}\nabla^\mu\chi\nabla_\mu\chi - 2\mathrm{e}^{4\phi}\Lambda \right] \right.$$
$$\left. + \mathrm{e}^{4\phi}\mathcal{L}_{\mathrm{matter}}\left(Q; \mathrm{e}^{2\phi}g^{(\mathrm{E})}\right) \right\}, \tag{2.28}$$

where f' should be regarded as a function of χ, $f'(\eta) = f'(\eta(\chi))$.

In case of the model in Eq. (2.9) with $\beta = 2$ as in (2.13), for the solution in Eq. (2.8) with $\eta_0 = \eta_1 = 0$ and in Eq. (2.10) with $\xi_0 = 0$ and ξ_1 given in (2.13), the condition (2.25) to avoid a ghost has the following form:

$$\frac{3}{4 + \frac{\Lambda}{3H_0^2 f_0}\mathrm{e}^{2H_0 t}} > 1, \tag{2.29}$$

which suggests the two cases. One is f_0 is positive but the bare cosmological constant Λ is negative and therefore $\xi_1 < -1$ from (2.13). Another is f_0 is negative and Λ is positive and therefore $\xi_1 > -1$. We note that in the model in Eq. (2.9) with $\beta = 2$ as in (2.13),

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$f'(\eta) = (f_0/2)\, \mathrm{e}^{\eta/2}$ and from a necessary condition in order to avoid a ghost $f'(\eta) > 0$ we find $f_0 > 0$, which corresponds to the former case. In this case, the condition (2.29) suggests that the de Sitter universe is stable in a period

$$\frac{1}{2H_0} \ln\left(-\frac{3H_0^2 f_0}{\Lambda}\right) < t < \frac{1}{2H_0}\left[\ln 4 + \ln\left(-\frac{3H_0^2 f_0}{\Lambda}\right)\right], \tag{2.30}$$

where we have also used the condition $(1 + f(\eta) - \xi)/6 > 0$ in (2.26). Hence the length of the ghost-free period is given by

$$\Delta t = \frac{\ln 4}{2H_0} = \frac{\ln 2}{H_0} \simeq \frac{0.69}{H_0}. \tag{2.31}$$

Thus unfortunately the period is less than one *e*-folding time. So this cannot give inflation in the early universe provided that the appearance of a ghost has to be avoided.

We note that if f_0 is negative and Λ is positive, from the condition (2.29) we see that the de Sitter universe is stable when $t > [1/(2H_0)]\,[\ln 4 + \ln(-3H_0^2 f_0/\Lambda)]$. However, this case cannot be available because the necessary condition in order to avoid a ghost $f'(\eta) = (f_0/2)\, \mathrm{e}^{\eta/2} > 0$ is not satisfied. We also remark that in Ref. [17], it has recently been shown that a de Sitter solution exists for an arbitrary value of ξ_0.

C. Cosmology in the Einstein frame

We explore cosmology in the Einstein frame and demonstrate that not only the non-phantom (quintessence) phase but also the phantom one can be realized in the framework of non-local gravity. We assume the FLRW metric (2.3), and consider the case when the contribution of matter is negligible.

In the absence of matter, the equations given by the variation of the action in Eq. (2.24) in terms of ϕ and η have the following forms:

$$0 = 12\left(\ddot{\phi} + 3H\dot{\phi}\right) + 2\left(\ddot{\eta} + 3H\dot{\eta}\right) - 2\mathrm{e}^{2\phi} f'(\eta)\dot{\eta}^2 + 8\mathrm{e}^{4\phi}\Lambda, \tag{2.32}$$

$$0 = 2\left(\ddot{\phi} + 3H\dot{\phi}\right) + 2\left(\frac{d}{dt} + 3H\right)\left(\mathrm{e}^{2\phi} f'(\eta)\dot{\eta}\right) - \mathrm{e}^{2\phi} f''(\eta)\dot{\eta}^2. \tag{2.33}$$

We also find that the first FLRW equation is given by

$$3H^2 = 3\dot{\phi}^2 + \dot{\phi}\dot{\eta} + \frac{\mathrm{e}^{2\phi}}{2} f'(\eta)\dot{\eta}^2 + \mathrm{e}^{4\phi}\Lambda. \tag{2.34}$$

We investigate the case that

$$f'(\eta) = \frac{f_0}{\beta} e^{\frac{\eta}{\beta}} . \tag{2.35}$$

We suppose

$$H = \frac{h_0}{t} , \quad \phi = -\frac{1}{2} \ln \frac{t}{t_0} , \quad \eta = \beta \ln \frac{t}{t_0} , \tag{2.36}$$

where h_0 and t_0 are constants. From the first equation in (2.36), we have $a = a_0 t^{h_0}$, where a_0 is a constant. In this case, Eqs. (2.32), (2.33) and (2.34) are reduced to the algebraic equations

$$0 = (-1 + 3h_0)(6 - 2\beta) - 2f_0\beta + 8t_0^2\Lambda , \tag{2.37}$$

$$0 = (-1 + 3h_0)(1 - 2f_0) - f_0 , \tag{2.38}$$

$$0 = -3h_0^2 + \frac{3}{4} + \frac{\beta}{2} - f_0\beta . \tag{2.39}$$

Then, we can solve f_0, β, and $t_0^2\Lambda$ as follows

$$f_0 = \frac{1 - 3h_0}{3(1 - 2h_0)} , \tag{2.40}$$

$$\beta = -\frac{9(1 - h_0)(3 - 4h_0)h_0}{2(1 - 9h_0 + 9h_0^2)} , \tag{2.41}$$

$$t_0^2\Lambda = \frac{3(-1 + 4h_0^2 - 9h_0^3 + 9h_0^4)}{4(1 - 9h_0 + 9h_0^2)} \tag{2.42}$$

Then arbitrary h_0, except the cases $h_0 = 1/2$ or $1 - 9h_0 + 9h_0^2 = 0$, that is, $h_0 = \frac{9 \pm 5\sqrt{3}}{18}$, can be realized by choosing f_0 and β properly if $\Lambda \neq 0$. Note that the value of Λ can be changed by the redefinition of t_0.

The effective equation of state for the universe is given by [8]

$$w_{\text{eff}} \equiv \frac{P_{\text{eff}}}{\rho_{\text{eff}}} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \frac{2}{3h_0} , \tag{2.43}$$

with

$$\rho_{\text{eff}} = \frac{3H^2}{\kappa^2} , \tag{2.44}$$

$$P_{\text{eff}} = -\frac{2\dot{H} + 3H^2}{\kappa^2} . \tag{2.45}$$

where ρ_{eff} and P_{eff} are the effective energy density and pressure of the universe, respectively. Moreover, in the last equality in Eq. (2.43), we have used the first equation in (2.36). If $\dot{H} = -h_0/t^2 < 0$, i.e., $h_0 > 0$, we have $w_{\text{eff}} > -1$, which represents the non-phantom

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(quintessence) phase, while for $\dot{H} = -h_0/t^2 > 0$, i.e., $h_0 < 0$, we find $w_{\text{eff}} < -1$, which describes the phantom phase. Since almost arbitrary h_0 can be realized, there can exist both cases of $h_0 > 0$ and $h_0 < 0$. We should note that Eq. (2.40) can be rewritten as follows,

$$h_0 = \frac{1 - 3f_0}{3(1 - 2f_0)}. \tag{2.46}$$

Therefore if we fix the model h_0 is uniquely determined.

We also remark that since we now consider the case in which the contribution of matter is absent, w_{eff} in Eq. (2.43) is equivalent to the equation of state for dark energy. This is because ρ_{eff} and P_{eff} correspond to ρ_{tot} and P_{tot}, respectively, where ρ_{tot} and P_{tot} are the total energy density and pressure of the universe, respectively, and thus w_{eff} can be expressed as $w_{\text{eff}} = P_{\text{tot}}/\rho_{\text{tot}}$.

Furthermore, since cosmological screening is a significant result of the paper, it will be useful to explain the mechanism in the Einstein frame. It follows from the form of the potential term for the scalar field ϕ in the action in Eq. (2.24) that ϕ runs towards negative infinity with time. Presumably, thereby screening the cosmological constant can be realized. It would be good to see this point explicitly. For clear understanding, we define the dimensionless quantities as $\tilde{\Lambda} \equiv t_0^2 \Lambda$ and $\tilde{H}_0 \equiv H(t = t_0)t_0 = h_0$, where the second equality follows from the first equation in (2.36). From the solution in Eq. (2.42), in principle, by solving Eq. (2.42) reversely, we obtain the expression of $\tilde{H}_0^2$ as a function of $\tilde{\Lambda}$ as

$$\tilde{H}_0^2 = \tilde{H}_0^2(\tilde{\Lambda}). \tag{2.47}$$

In the second equation $\phi = -(1/2)\ln(t/t_0)$ in (2.24), $\phi = 0$ at $t = t_0$. In other words, the definition in terms of the value of t_0 can be determined by the dynamics (or expression) of ϕ. Thus, it can be considered that Eq. (2.47) implies that the cosmological constant Λ is effectively screened by ϕ, similarly to that in Eq. (2.15) in the Jordan frame. Incidentally, a similar mechanism was proposed in Ref. [25] to address the hierarchy problem.

In addition, there are strong experimental constraints on violation of equivalence principle and the existence of a fifth force mediated by scalars, which is easily seen in the Einstein frame. In the non-local model, the scalar field ϕ couples to the matter sector and this could be a serious problem. Essentially, what needs to be very small is $(d\Omega/dt)/\Omega$. It is necessary to check this before presenting the model as a serious candidate for dark energy. For the solution $\phi = -(1/2)\ln(t/t_0)$ in the second equation in (2.24), we find $(d\Omega/dt)/\Omega = \dot{\phi} = -1/(2t)$.

It is clear that in the limit of $t \to \infty$, $\dot{\phi} \to 0$. For example, this limit can be regarded as the present time. Consequently, the rate of the time variation of Ω can be very small at the current universe.

D.   Addition of an R^2 term

We examine the influence of adding an R^2 term on the stability of non-local gravity in the Einstein frame. The addition of it is motivated by the possible curing of the finite-time future singularities. We study the case in which $uR^2/(2\kappa^2)$ with $u(\neq 0)$ being a non-zero arbitrary constant is added to the action in Eq. (2.1) as follows:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\Box^{-1}R) \right) + uR^2 - 2\Lambda \right] + \mathcal{L}_{\mathrm{matter}} \left(Q; g \right) \right\} . \qquad (2.48)$$

We introduce another scalar field ζ. The action in Eq. (2.48) can be rewritten to

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\eta) \right) - \partial_\mu \xi \partial^\mu \eta - \xi R + u \left(2\zeta R - \zeta^2 \right) - 2\Lambda \right] + \mathcal{L}_{\mathrm{matter}} \right\} . \qquad (2.49)$$

By varying the action in Eq. (2.49) with respect to ζ, we have $\zeta = R$. Substituting this equation into the action in Eq. (2.49), the action in Eq. (2.48) is re-obtained.

When an R^2 term is added, i.e., for the action in Eq. (2.48), in the flat FLRW background in Eq. (2.3), gravitational field equations are explicitly written down as

$$0 = -3H^2 \left(1 + f(\eta) - \xi \right) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H \left(f'(\eta)\dot{\eta} - \dot{\xi} \right) + \Theta + \Lambda + \kappa^2 \rho_{\mathrm{m}} , \qquad (2.50)$$

$$0 = \left(2\dot{H} + 3H^2 \right) \left(1 + f(\eta) - \xi \right) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt} \right) \left(f(\eta) - \xi \right) + \Xi - \Lambda + \kappa^2 P_{\mathrm{m}} , \qquad (2.51)$$

where Θ and Ξ are the contributions from the additional term $uR^2/(2\kappa^2)$, given by

$$\Theta \equiv u \left(-6H^2 R + \frac{1}{2}R^2 - 6H\dot{R} \right) = 18u \left(-6H^2\dot{H} + \dot{H}^2 - 2H\ddot{H} \right) , \qquad (2.52)$$

$$\Xi \equiv u \left[2 \left(2\dot{H} + 3H^2 \right) R - \frac{1}{2}R^2 + 2\ddot{R} + 4H\dot{R} \right]$$

$$= 6u \left(9\dot{H}^2 + 18H^2\dot{H} + 2\dddot{H} + 12H\ddot{H} \right) . \qquad (2.53)$$

Namely, Θ is added to the r.h.s. of Eq. (2.4) and Ξ is added to the r.h.s. of Eq. (2.5). Here, in deriving the second equalities in Eqs. (2.52) and (2.53) we have used $R = 6\dot{H} + 12H^2$. From Eqs. (2.52) and (2.53), we see that all the terms in Θ and Ξ include the derivative

terms of H with respect to time. Hence, for a de Sitter solution $H = H_0$, Θ and Ξ vanish and therefore do not change the de Sitter solutions. As a result, the solutions in (2.14) and (2.17) can also be obtained in the presence of the additional R^2 term.

By using Eqs. (2.4), (2.5), (2.43), (2.44) and (2.45), the effective equation of state for the universe is given by

$$w_{\text{eff}} = \frac{P_{\text{eff}}}{\rho_{\text{eff}}} = \frac{\left(2\dot{H} + 3H^2\right)(f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)(f(\eta) - \xi) - \Lambda + \kappa^2 P_{\text{m}}}{-3H^2(f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Lambda + \kappa^2 \rho_{\text{m}}},$$

(2.54)

where

$$\rho_{\text{eff}} = \frac{1}{\kappa^2}\left[-3H^2(f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Lambda + \kappa^2 \rho_{\text{m}}\right],$$

(2.55)

$$P_{\text{eff}} = \frac{1}{\kappa^2}\left[\left(2\dot{H} + 3H^2\right)(f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)(f(\eta) - \xi) - \Lambda + \kappa^2 P_{\text{m}}\right].$$

(2.56)

If we add an R^2 term as in the action in Eq. (2.48), ρ_{eff} and P_{eff} become

$$\rho_{\text{eff}} = \frac{1}{\kappa^2}\left[-3H^2(f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Theta + \Lambda + \kappa^2 \rho_{\text{m}}\right],$$

(2.57)

$$P_{\text{eff}} = \frac{1}{\kappa^2}\left[\left(2\dot{H} + 3H^2\right)(f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)(f(\eta) - \xi)\right.$$
$$\left. + \Xi - \Lambda + \kappa^2 P_{\text{m}}\right].$$

(2.58)

For a de Sitter solution $H = H_0$, since Θ and Ξ vanish, w_{eff} does not change in the presence of an R^2 term. In Sec. V C, we discuss the cosmological consequences of adding an R^2 term for more general solutions. In particular, it is shown that an R^2 term plays a fundamental role to cure the finite-time future singularities in non-local gravity.

Let us examine the condition to avoid a ghost in the present case. By following the same procedure in Sec. II C, we perform a conformal transformation to the Einstein frame as in (2.18) with

$$\Omega^2 = \frac{1}{1 + f(\eta) - \xi + 2u\zeta}.$$

(2.59)

In this case, we find

$$S = \int d^4x \sqrt{-g^{(\text{E})}}\left[\frac{1}{2\kappa^2}\left(R^{(\text{E})} - 6g^{(\text{E})\,\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - e^{2\phi}g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\eta - ue^{4\phi}\zeta^2 - 2e^{4\phi}\Lambda\right)\right.$$
$$\left. + e^{4\phi}\mathcal{L}_{\text{matter}}\left(Q; e^{2\phi}g^{(\text{E})}\right)\right],$$

(2.60)

where $\phi = \ln \Omega = -(1/2) \ln (1 + f(\eta) - \xi + 2u\zeta)$. Substituting $\xi = -e^{-2\phi} + (1 + f(\eta)) + 2u\zeta$ into the action in Eq. (2.60), we acquire

$$S = \int d^4x \sqrt{-g^{(\mathrm{E})}} \left[\frac{1}{2\kappa^2} \left(R^{(\mathrm{E})} - 6\nabla^\mu \phi \nabla_\mu \phi - 2\nabla^\mu \phi \nabla_\mu \eta - e^{2\phi} f'(\eta) \nabla^\mu \eta \nabla_\mu \eta \right. \right.$$
$$\left. \left. -2u e^{2\phi} \nabla^\mu \zeta \nabla_\mu \eta - u e^{4\phi} \zeta^2 - 2e^{4\phi} \Lambda \right) + e^{4\phi} \mathcal{L}_{\mathrm{matter}} \left(Q; e^{2\phi} g^{(\mathrm{E})} \right) \right] . \tag{2.61}$$

In this case, the matrix for the kinetic terms is given by

$$M \equiv \begin{pmatrix} 6 & 1 & u e^{2\phi} \\ 1 & e^{2\phi} f'(\eta) & 0 \\ u e^{2\phi} & 0 & 0 \end{pmatrix} . \tag{2.62}$$

The necessary condition to avoid a ghost is that all the eigenvalues of the matrix for the kinetic terms M must be positive. The characteristic equation for M is given by

$$\det |M - yE| = \det \begin{vmatrix} 6 - y & 1 & u e^{2\phi} \\ 1 & e^{2\phi} f'(\eta) - y & 0 \\ u e^{2\phi} & 0 & -y \end{vmatrix} = 0 , \tag{2.63}$$

where y denotes an eigenvalue of M and E is a unit matrix. From Eq. (2.63), we obtain

$$y^3 - Y_1 y^2 + Y_2 y - Y_3 = 0 , \tag{2.64}$$

with

$$Y_1 \equiv y_1 + y_2 + y_3 = e^{2\phi} f'(\eta) + 6 , \tag{2.65}$$
$$Y_2 \equiv y_2 y_3 + y_3 y_1 + y_1 y_2 = 6 e^{2\phi} f'(\eta) - u^2 e^{4\phi} - 1 , \tag{2.66}$$
$$Y_3 \equiv y_1 y_2 y_3 = -u^2 e^{6\phi} f'(\eta) , \tag{2.67}$$

where y_1, y_2 and y_3 are three eigenvalues of M. In order to avoid a ghost, we must have $y_1 > 0$, $y_2 > 0$ and $y_3 > 0$, and therefore $Y_1 > 0$, $Y_2 > 0$ and $Y_3 > 0$. If $f'(\eta) < 0$, $Y_3 > 0$. Moreover, if $-6e^{-2\phi} < f'(\eta)$, $Y_1 > 0$. However, when $f'(\eta) < 0$, $Y_2 < 0$. This implies that all the three eigenvalues of M cannot be positive simultaneously. Thus, unfortunately the necessary condition to avoid a ghost cannot be satisfied. It may be noted, however, that the addition of a non-minimally coupled scalar instead of an R^2 term may help in the realization of a solution without a ghost [26]. We note that the conditions $Y_1 > 0$ and $Y_3 > 0$ correspond to $\mathrm{Tr}\, M > 0$ and $\det |M| > 0$, respectively.

If we add the general $F(R)$ term instead of a simple R^2 term and, as usual, by rewriting the action as [8] $F(R)$ to $F(A) + F'(A)(R - A)$, where A is an auxiliary field and $F'(A) \equiv dF(A)/dA$, we may redo the analysis parallel to the present R^2 case. Clearly, however, if we introduce a scalar field ζ by setting $\zeta = F'(A)/(2u)$, the ghost-free condition does not change at all. Hence the addition of an $F(R)$ term does not remove the ghost. Also note that the previous de Sitter solution would no longer be a solution.

We note that the creation of the Standard Model particles is very important and serious problem. Since the non-local gravity is very complicated model, however, it is very important to show the existence of the consistent de Sitter solution, which could be applied to the cosmic accelerated expansion in the early universe, i.e., inflation. In order to generate radiation and non-relativistic matters including Standard Model quarks and leptons, it is necessary to consider a kind of the existence of the inflaton field, which decays into radiation, as a standard inflation scenario, and a scenario for the baryogenesis and/or leptogenesis as well as a generation mechanism of dark matter. These problems of the particle creations consistent with the observation of the universe could be investigated in the future works.

We also mention that gravity being universal will couple to visible and hidden sectors, and in action Eq. (2.49) the couplings will be universal. Therefore, it is important to consider how we could explain that after inflation solely the Standard Model degrees are excited from the vacuum. In addition, Big Bang Nucleosynthesis (BBN) puts stringent constraint on any hidden or dark radiation. Our discussion in Sec. II E is an attempt to examine the influence of adding an R^2 term on the stability of non-local gravity in the Einstein frame, i.e., the ghost-free condition. Thus, it is necessary not only to explore a mechanism to create the Standard Model degrees after inflation but also to investigate whether the resultant inflationary scenario in non-local gravity could satisfy the BBN constraints. We like to investigate these problems in future.

III. NON-LOCAL GRAVITY WITH LAGRANGE CONSTRAINT MULTIPLIER

Recently, in Refs. [27, 28] the model of covariant gravity with properties similar to Hořava gravity [29] was introduced, adding Lagrange constraint multiplier [30]. In this section, we study the Hořava like extension covariantly proposed in Ref. [28]. We generalize non-local gravity by introducing Lagrange constraint multiplier and examine a de Sitter solution in

non-local gravity with Lagrange constraint multiplier [13].

In this section, however, we do not discuss the condition for the absence of a ghost. This is partly because the analysis would become much more complicated. But the main reason is that in the present case the local Lorentz invariance will be explicitly violated in all solutions of the theory. This implies that the existence of a ghost, which would be fatal for a quatum theory with Lorentz invariance, may not be a real problem at all.

The constrained action for a scalar field ψ is given by

$$ S_\psi = \int d^4x \sqrt{-g} \left[-\lambda \left(\frac{1}{2} \partial_\mu \psi \partial^\mu \psi + U(\psi) \right) \right] , \tag{3.1} $$

where λ is the Lagrange multiplier field. This action gives a constraint $(1/2)\, \partial_\mu \psi \partial^\mu \psi + U(\psi) = 0$. The vector $\partial_\mu \psi$ is time-like. Therefore, the direction of time can be chosen to be parallel to $\partial_\mu \psi$ at least locally. Hence, it follows from this constraint that $(1/2)\, (d\psi/dt)^2 = U(\psi)$. For simplicity, we choose $U(\psi)$ to be a constant as $U(\psi) = U_0$. In this case, we have

$$ \frac{1}{2} \partial_\mu \psi \partial^\mu \psi + U_0 = 0 . \tag{3.2} $$

Under the constraint (3.2), we now define

$$ R^{(2n+2)} \equiv R - 2\kappa^2 \alpha \left[(\partial^\mu \psi \partial^\nu \psi \nabla_\mu \nabla_\nu + 2 U_0 \nabla^\rho \nabla_\rho)^n \left(\partial^\mu \psi \partial^\nu \psi R_{\mu\nu} + U_0 R \right) \right]^2 , $$

$$ R^{(2n+3)} \equiv R - 2\kappa^2 \alpha \left[(\partial^\mu \psi \partial^\nu \psi \nabla_\mu \nabla_\nu + 2 U_0 \nabla^\rho \nabla_\rho)^n \left(\partial^\mu \psi \partial^\nu \psi R_{\mu\nu} + U_0 R \right) \right] $$

$$ \times \left[(\partial^\mu \psi \partial^\nu \psi \nabla_\mu \nabla_\nu + 2 U_0 \nabla^\rho \nabla_\rho)^{n+1} \left(\partial^\mu \psi \partial^\nu \psi R_{\mu\nu} + U_0 R \right) \right] , $$

$$ \Box^{(n)} \equiv \Box + \gamma \left(\partial^\mu \psi \partial^\nu \psi \nabla_\mu \nabla_\nu + 2 U_0 \nabla^\rho \nabla_\rho \right)^n , \tag{3.3} $$

where n, α and γ are constants. In a parallel way from Eq. (2.1), we may define the non-local action [13]

$$ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R^{(m)} \left(1 + f \left((\Box^n)^{-1} R^{(m)} \right) \right) - 2\Lambda \right] \right. $$

$$ \left. -\lambda \left(\frac{1}{2} \partial_\mu \psi \partial^\mu \psi + U_0 \right) + \mathcal{L}_{\text{matter}} \right\} , \tag{3.4} $$

and rewrite the action in Eq. (3.4) in a local way by introducing two scalar fields η and ξ [13]:

$$ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R^{(m)} \left(1 + f(\eta) \right) + \xi \left(\Box^{(n)} \eta - R^{(m)} \right) - 2\Lambda \right] - \lambda \left(\frac{1}{2} \partial_\mu \psi \partial^\mu \psi + U_0 \right) \right. $$

$$ \left. + \mathcal{L}_{\text{matter}} \right\} . \tag{3.5} $$

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In Eqs. (3.4) and (3.5), n can be even or odd integer.

In order to explore the cosmology, we assume the following form of the metric:

$$ds^2 = -e^{2b(t)} dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i \right)^2 , \tag{3.6}$$

and that the scalar field ψ only depends on time. Then, we have

$$\partial^\mu \psi \partial^\nu \psi R_{\mu\nu} + U_0 R = 6U_0 e^{-2b} H^2 , \quad \partial^\mu \psi \partial^\nu \psi \nabla_\mu \nabla_\nu + 2U_0 \nabla^\rho \nabla_\rho = -6U_0 e^{-2b} H \partial_t ,$$
$$R = e^{-2b} \left(6\dot{H} + 12H^2 - 6\dot{b}H \right) . \tag{3.7}$$

We suppose the scalar fields η and ξ only depend on t. We examine most simple but non-trivial case that $m = 2$, i.e., $n = 0$, in Eq. (3.5) and $\gamma = 0$ in (3.3). For $m = 2$, it follows from the first equation in (3.3) with $n = 0$ that

$$R^{(2)} = R - 2\kappa^2 \alpha \left(\partial^\mu \psi \partial^\nu \psi R_{\mu\nu} + U_0 R \right)^2 . \tag{3.8}$$

The variation of the action in Eq. (3.5) with respect to η gives $0 = \Box\xi + f'(\eta)R^{(2)}$. In the background in Eq. (3.6), this equation yields

$$0 = \left(6\dot{H} + 12H^2 - 72\kappa^2 \alpha U_0^2 H^4 \right) f'(\eta) - \ddot{\xi} - 3H\dot{\xi} , \tag{3.9}$$

after putting $b = 0$. Moreover, the variation of the action in Eq. (3.5) with respect to ξ leads to $\Box\eta = R^{(2)}$. In the background in Eq. (3.6), this equation becomes

$$0 = 6\dot{H} + 12H^2 - 72\kappa^2 \alpha U_0^2 H^4 + \ddot{\eta} + 3H\dot{\eta} . \tag{3.10}$$

In deriving Eqs. (3.9) and (3.10), we have used Eq. (3.8) and the first and third equations in (3.7).

For $m = 2$, the action in Eq. (3.5) is expressed as

$$S = \int d^4x a^3 \left(\frac{1}{2\kappa^2} \left\{ \left[e^{-b} \left(6\dot{H} + 12H^2 - 6\dot{b}H \right) - 72\kappa^2 \alpha U_0^2 e^{-3b} H^4 \right] (1 + f(\eta) - \xi) \right. \right.$$
$$\left. \left. + e^{-b}\dot{\xi}\dot{\eta} - 2e^b\Lambda \right\} - \lambda \left(-\frac{e^{-b}}{2}\dot{\psi}^2 + e^b U_0 \right) + \mathcal{L}_{\text{matter}} \right) , \tag{3.11}$$

where we have used the metric in Eq. (3.6), the first and third equations in (3.7) and Eq. (3.8).

By the variation of the action in Eq. (3.11) with respect to b, we obtain the equation corresponding to the first FLRW equation:

$$0 = -3 \left(H^2 + 36\kappa^2 \alpha U_0^2 H^4 \right) (1 + f(\eta) - \xi) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H \left(f'(\eta)\dot{\eta} - \dot{\xi} \right) + \Lambda + 2\kappa^2 \lambda U_0 + \kappa^2 \rho_{\text{m}} , \tag{3.12}$$

where we have used $(1/2)\left(d\psi/dt\right)^2 = U_0$. Moreover, the variation of the action in Eq. (3.11) with respect to a gives the equation corresponding to the second FLRW equation:

$$
\begin{aligned}
0 = {} & \left(2\dot{H} + 3H^2 + 108\kappa^2\alpha U_0^2 H^4 + 144\kappa^2\alpha U_0^2 H^2\dot{H}\right)\left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} \\
& + \left(2H + 48\kappa^2\alpha U_0^2 H^3\right)\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + f''(\eta)\left(\dot{\eta}\right)^2 + f'(\eta)\ddot{\eta} - \ddot{\xi} - \Lambda + \kappa^2 P_{\mathrm{m}}\,.
\end{aligned}
\tag{3.13}
$$

We investigate if there could be a solution describing the de Sitter space. When we assume H is a constant, $H = H_0$, Eq. (3.10) could be integrated, as executed in (2.8),

$$
\eta = -4H_0\left(1 - 6\kappa^2\alpha U_0^2 H_0^2\right)t - \eta_0 \mathrm{e}^{-3H_0 t} + \eta_1\,.
\tag{3.14}
$$

Again we take $\eta_0 = \eta_1 = 0$. We suppose $f(\eta)$ is given by (2.9) as $f(\eta) = f_0 \mathrm{e}^{\frac{\eta}{\beta}}$. By integrating Eq. (3.9), we obtain

$$
\xi = -\frac{3f_0 H_0 t_{\mathrm{c}}}{1 + 3H_0 t_{\mathrm{c}}}\mathrm{e}^{\frac{t}{t_{\mathrm{c}}}} + \frac{\xi_0}{3H_0}\mathrm{e}^{-3H_0 t} - \xi_1\,.
\tag{3.15}
$$

Here, ξ_0 and ξ_1 are constants of the integration. Moreover, t_{c} is given by

$$
t_{\mathrm{c}} \equiv -\frac{\beta}{4H_0\left(1 - 6\kappa^2\alpha U_0^2 H_0^2\right)}\,.
\tag{3.16}
$$

We now neglect the contribution from matter. By substituting Eqs. (3.14) with $\eta_0 = \eta_1 = 0$ and (3.15) into Eq. (3.13), we find

$$
\xi_0 = 0\,, \quad \xi_1 = -1 + \frac{\Lambda}{3\left(H_0^2 + 36\kappa^2\alpha U_0^2 H_0^4\right)}\,,
\tag{3.17}
$$

and

$$
0 = 18\left(1 + 36\kappa^2\alpha U_0^2 H_0^2\right)H_0^3 t_{\mathrm{c}}^3 + 3\left(7 + 120\kappa^2\alpha U_0^2 H_0^2\right)H_0^2 t_{\mathrm{c}}^2 + 8\left(1 + 6\kappa^2\alpha U_0^2 H_0^2\right)H_0 t_{\mathrm{c}} + 1\,,
\tag{3.18}
$$

which is the cubic algebraic equation with respect to t_{c}. Hence, there is always a real solution of t_{c}. Thus, we can find the value of β by using Eq. (3.16). Consequently, if we choose β properly, there always appears the solution describing the de Sitter universe.

The second equation in (3.17) can be rewritten as

$$
H_0^2 = \frac{\Lambda}{3\left(1 + \xi_1\right)} - 36\kappa^2\alpha U_0^2 H_0^4\,.
\tag{3.19}
$$

Therefore, ξ_1 can be a screening of the cosmological constant and the last term is a correction coming from the covariant Hořava like model.

As discussed in Sec. II E, the non-local action often appears as a quantum correction. Equation (3.19) implies again that the correction effectively changes the value of the cosmological constant. If $\xi \sim 0$ in the early universe, where $t \sim 0$, Eq. (3.15) means

$$\xi_1 \sim -\frac{3f_0 H_0 t_c}{1 + 3H_0 t_c} \, . \tag{3.20}$$

If H_0 corresponds to the value of the Hubble parameter in the present universe, the second term on the right-hand side (r.h.s.) of Eq. (3.19) could be negligible. Thus, if $-\frac{3f_0 H_0 t_c}{1+3H_0 t_c}$ is positive and very large, the effective cosmological constant in the present universe could be very small.

We explore a condition to avoid the appearance of a ghost by following the procedure in Sec. II C. We make a conformal transformation in (2.18) to the Einstein frame:

$$g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(\mathrm{E})} , \quad R^{(2)} = \frac{1}{\Omega^2} \left[R^{(2,\mathrm{E})} - 6 \left(\Box \ln \Omega + g^{(\mathrm{E}) \, \mu\nu} \nabla_\mu \ln \Omega \nabla_\nu \ln \Omega \right) \right] , \tag{3.21}$$

with Eq. (2.19). We here describe the Lagrangian for the part of the Lagrange multiplier field as

$$\mathcal{L}_\lambda \left(\Upsilon ; g \right) = -\lambda \left(\frac{1}{2} \partial_\mu \psi \partial^\mu \psi + U_0 \right) , \tag{3.22}$$

where Υ denotes the Lagrange multiplier field λ and the scalar field ψ. Through the conformal transformation in (3.21), the action in the Einstein frame is expressed as

$$S = \int d^4 x \sqrt{-g^{(\mathrm{E})}} \left[\frac{1}{2\kappa^2} \left(R^{(2,\mathrm{E})} - 6\nabla^\mu \phi \nabla_\mu \phi - 2\nabla^\mu \phi \nabla_\mu \eta - \mathrm{e}^{2\phi} f'(\eta) \nabla^\mu \eta \nabla_\mu \eta - 2\mathrm{e}^{4\phi} \Lambda \right) \right.$$
$$\left. + \mathrm{e}^{4\phi} \mathcal{L}_\lambda \left(\Upsilon ; \mathrm{e}^{2\phi} g^{(\mathrm{E})} \right) + \mathrm{e}^{4\phi} \mathcal{L}_{\mathrm{matter}} \left(Q ; \mathrm{e}^{2\phi} g^{(\mathrm{E})} \right) \right] . \tag{3.23}$$

Since the form of the action in Eq. (3.23) is equivalent to that in Eq. (2.24) except the Lagrangian for the part of the Lagrange multiplier field $\mathcal{L}_\lambda$, a condition to avoid a ghost is given by (2.25). We also have a necessary condition $f'(\eta) > 0$. Hence, the condition to be free of ghost is given by Eq. (2.26).

In case of the model in Eq. (2.9), for the solution in Eq. (3.14) with $\eta_0 = \eta_1 = 0$ and in Eq. (3.15) with $\xi_0 = 0$ and ξ_1 given in (3.17), the condition (2.25) to avoid a ghost is described by the following form:

$$\frac{6}{\beta \left\{ 1 - \frac{3\beta}{4\left(1 - 6\kappa^2 \alpha U_0^2 H_0^2\right) - 3\beta} + \frac{\Lambda}{3f_0 H_0^2 \left(1 + 36\kappa^2 \alpha U_0^2 H_0^2\right)} \exp\left[\frac{4H_0\left(1 - 6\kappa^2 \alpha U_0^2 H_0^2\right)}{\beta} t \right] \right\}} > 1 \, . \tag{3.24}$$

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In addition, from a necessary condition $f'(\eta) > 0$ we find $f_0/\beta > 0$, which implies that the sign of f_0 is the same as that of β. In the same way, a de Sitter solution of arbitrary non-local gravity with Lagrange constraint multiplier may be found. Preliminary consideration shows that adding an R^2 term may also make a screening scenario to be realistic enough.

IV. FINITE-TIME FUTURE SINGULARITIES IN NON-LOCAL GRAVITY

In this section, we examine whether there exists the finite-time future singularities in non-local gravity. In the flat FLRW space-time, we analyze an asymptotic solution of the gravitational field equations (2.4) and (2.5) in the limit of the time when the finite-time future singularities appear.

A. Finite-time future singularities

We consider the case in which the Hubble parameter H is expressed as

$$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q} \,, \tag{4.1}$$

where h_{s} is a positive constant, q is a non-zero constant larger than -1 ($q > -1, q \neq 0$), and t_{s} is the time when the finite-time future singularity appears. We only consider the period $0 < t < t_{\mathrm{s}}$ because H should be real number. When $t \to t_{\mathrm{s}}$, for $q > 1$ and $0 < q < 1$, $H \sim h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-q}$ as well as $\dot{H} \sim q h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-(q+1)}$ become infinity and hence the scalar curvature R diverges. For $-1 < q < 0$, H is finite, but $\dot{H}$ becomes infinity and therefore R also diverges.

From Eq. (4.1), we obtain

$$a \sim a_{\mathrm{s}} \exp\left[\frac{h_{\mathrm{s}}}{q-1} (t_{\mathrm{s}} - t)^{-(q-1)} \right] \,, \tag{4.2}$$

where a_{s} is a constant. By using $\ddot{\eta} + 3H\dot{\eta} = a^{-3} d\left(a^3 \dot{\eta}\right)/dt$ and Eq. (2.6), η is described as

$$\eta = -\int^t \frac{1}{a^3} \left(\int^{\bar{t}} R a^3 d\bar{t} \right) dt \,. \tag{4.3}$$

In the limit $t \to t_{\mathrm{s}}$, for $q > 1$, $\dot{H} \ll H^2$ and hence $R \sim 12H^2$, whereas for $-1 < q < 0$ and $0 < q < 1$, $\dot{H} \gg H^2$ and hence $R \sim 6\dot{H}$. By applying these relations to Eq. (4.3) and

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taking the leading term in terms of $(t_s - t)$, we obtain

$$\eta \sim -\frac{4h_s}{q-1}(t_s - t)^{-(q-1)} + \eta_c \quad (q > 1) , \tag{4.4}$$

$$\eta \sim -\frac{6h_s}{q-1}(t_s - t)^{-(q-1)} + \eta_c \quad (-1 < q < 0 , \; 0 < q < 1) , \tag{4.5}$$

where η_c is an integration constant. We describe the derivation of Eqs. (4.4) and (4.5) from Eq. (4.3) in Appendix A.

B. Analysis for $\eta_c \neq 0$

We study the case of $\eta_c \neq 0$. (In the next subsection, we explore the case of $\eta_c = 0$.) If the power in terms of $(t_s - t)$ is negative (positive), $-(q-1) < 0 \; (> 0)$, the first term proportional to $(t_s - t)^{-(q-1)}$ (the second constant term) is the leading one. Hence, for $q > 1$, the first term is the leading one, i.e., $\eta \propto (t_s - t)^{-(q-1)}$, while for $-1 < q < 0$ and $0 < q < 1$, the second term is the leading one, i.e., $\eta \sim \eta_c$. We note that if $q = 1$ in Eq. (4.1), it follows from Eq. (2.6) that $\eta \sim 6h_s \left[(1 + 2h_s)/(1 + 3h_s)\right] \ln (t_s - t) + \eta_c$.

We take a form of $f(\eta)$ as

$$f(\eta) = f_s \eta^\sigma , \tag{4.6}$$

where $f_s (\neq 0)$ and $\sigma (\neq 0)$ are non-zero constants. By using $\ddot{\xi} + 3H\dot{\xi} = a^{-3} d \left(a^3 \dot{\xi} \right)/dt$ and Eq. (2.7), ξ is written as

$$\xi = \int^t \frac{1}{a^3} \left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} R a^3 d\bar{t} \right) dt . \tag{4.7}$$

By applying $R \sim 12H^2$ (for $q > 1$) and $R \sim 6\dot{H}$ (for $q < 1$) to Eq. (4.7) and taking the leading term in terms of $(t_s - t)$, we acquire

$$\xi \sim -f_s \left(-\frac{4h_s}{q-1} \right)^\sigma (t_s - t)^{-(q-1)\sigma} + \xi_c \quad (q > 1) , \tag{4.8}$$

$$\xi \sim \frac{6f_s h_s \sigma \eta_c^{\sigma-1}}{q-1}(t_s - t)^{-(q-1)} + \xi_c \quad (-1 < q < 0 , \; 0 < q < 1) , \tag{4.9}$$

where ξ_c is an integration constant.

If the power in terms of $(t_s - t)$ is negative (positive), $-(q-1)\sigma < 0 \; (> 0)$, the first term proportional to $(t_s - t)^{-(q-1)\sigma}$ (the second constant term) is the leading one. Therefore, for $[q > 1, \; \sigma > 0]$, $\xi \propto (t_s - t)^{-(q-1)\sigma}$, whereas for $[q > 1, \; \sigma < 0]$ and $[-1 < q < 0 , \; 0 < q < 1]$, $\xi \sim \xi_c$.

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Thus, there are three cases. (i) $[q > 1, \sigma > 0]$: $\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$ and $\xi \propto (t_{\mathrm{s}} - t)^{-(q-1)\sigma}$. (ii) $[q > 1, \sigma < 0]$: $\eta \propto (t_{\mathrm{s}} - t)^{-(q-1)}$ and $\xi \sim \xi_{\mathrm{c}}$. (iii) $[-1 < q < 0\,, 0 < q < 1]$: $\eta \sim \eta_{\mathrm{c}}$ and $\xi \sim \xi_{\mathrm{c}}$.

We examine the behavior of each term on the r.h.s. of Eqs. (2.4) and (2.5) in the limit $t \to t_{\mathrm{s}}$, in particular that of the leading terms, and study the condition that an asymptotic solution can be obtained. When $t \to t_{\mathrm{s}}$, Λ, ρ_{m} and P_{m} can be neglected because these values are finite.

First, we investigate each term on the r.h.s. of Eq. (2.4). For case (i) $[q > 1, \sigma > 0]$, the first, second and third terms evolve as

$$-3H^2 \left(1 + f(\eta) - \xi\right) \sim -3H^2 \left(f(\eta) - \xi\right) \sim -6h_{\mathrm{s}}^2 f_{\mathrm{s}} \left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma} (t_{\mathrm{s}} - t)^{-[(q-1)\sigma + 2q]} \,, \quad (4.10)$$

$$\frac{1}{2}\dot{\xi}\dot{\eta} \sim -8h_{\mathrm{s}}^2 \sigma f_{\mathrm{s}} \left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma-1} (t_{\mathrm{s}} - t)^{-[(q-1)\sigma + q + 1]} \,, \quad (4.11)$$

$$-3H \left(\frac{df(\eta)}{d\eta}\dot{\eta} - \dot{\xi}\right) \sim 24h_{\mathrm{s}}^2 \sigma f_{\mathrm{s}} \left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma-1} (t_{\mathrm{s}} - t)^{-[(q-1)\sigma + q + 1]} \,. \quad (4.12)$$

In the first relation in (4.10), we have used $1 \ll |f(\eta) - \xi|$. In this case, the first term is proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma + 2q]}$, whereas the second and third terms are proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma + q + 1]}$. Hence, the first term is the leading one. The coefficient of the first term cannot be zero because $h_{\mathrm{s}} \neq 0$ and $f_{\mathrm{s}} \neq 0$ and hence the leading term cannot vanish. This implies that in this case, $H \sim h_{\mathrm{s}}/(t_{\mathrm{s}} - t)^q$ in Eq. (4.1) cannot be a solution describing the finite-time future singularities.

For case (ii) $[q > 1, \sigma < 0]$, the first term evolves as

$$-3H^2 \left(1 + f(\eta) - \xi\right) \sim -3H^2 \left(1 - \xi_{\mathrm{c}}\right) \sim -3h_{\mathrm{s}}^2 \left(1 - \xi_{\mathrm{c}}\right) (t_{\mathrm{s}} - t)^{-2q} \,, \quad (4.13)$$

where in the first relation, we have used $f(\eta) \sim 0$. The second and third terms are given by Eqs. (4.11) and (4.12), respectively. In this case, the first term is proportional to $(t_{\mathrm{s}} - t)^{-2q}$, whereas the second and third terms are proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma + q + 1]}$. Hence, the first term is the leading one. From Eq. (4.13), we find the condition that the first term vanishes is given by

$$\xi_{\mathrm{c}} = 1 \,. \quad (4.14)$$

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For case (iii) $[-1 < q < 0\,,\ 0 < q < 1]$, the first, second and third terms evolve as

$$-3H^2\left(1 + f(\eta) - \xi\right) \sim -3h_{\mathrm{s}}^2\left(1 + f_{\mathrm{s}}\eta_{\mathrm{c}}^{\sigma} - \xi_{\mathrm{c}}\right)(t_{\mathrm{s}} - t)^{-2q}\,, \tag{4.15}$$

$$\frac{1}{2}\dot{\xi}\dot{\eta} \sim -18h_{\mathrm{s}}^2\sigma f_{\mathrm{s}}\eta_{\mathrm{c}}^{\sigma-1}(t_{\mathrm{s}} - t)^{-2q}\,, \tag{4.16}$$

$$-3H\left(\frac{df(\eta)}{d\eta}\dot{\eta} - \dot{\xi}\right) \sim 36h_{\mathrm{s}}^2\sigma f_{\mathrm{s}}\eta_{\mathrm{c}}^{\sigma-1}(t_{\mathrm{s}} - t)^{-2q}\,. \tag{4.17}$$

These terms are proportional to $(t_{\mathrm{s}} - t)^{-2q}$, which are the leading ones. Substituting Eqs. (4.15), (4.16) and (4.17) into Eq. (2.4), the condition that these leading terms vanish is given by

$$f_{\mathrm{s}}\eta_{\mathrm{c}}^{\sigma-1}\left(6\sigma - \eta_{\mathrm{c}}\right) + \xi_{\mathrm{c}} - 1 = 0\,. \tag{4.18}$$

Next, we investigate the r.h.s. of Eq. (2.5). For case (i) $[q > 1,\ \sigma > 0]$, the first, third and fourth terms evolve as

$$\left(2\dot{H} + 3H^2\right)\left(1 + f(\eta) - \xi\right) \sim 3H^2\left(1 + f(\eta) - \xi\right) \sim 3H^2\left(f(\eta) - \xi\right)$$
$$\sim 6h_{\mathrm{s}}^2 f_{\mathrm{s}}\left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma}(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2q]}\,, \tag{4.19}$$

$$\frac{d^2\left(f(\eta) - \xi\right)}{dt^2} \sim -8h_{\mathrm{s}}f_{\mathrm{s}}\sigma\left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma-1}\left[(q-1)\sigma + 1\right](t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2]}\,, \tag{4.20}$$

$$2H\frac{d\left(f(\eta) - \xi\right)}{dt} \sim -16h_{\mathrm{s}}^2 f_{\mathrm{s}}\sigma\left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma-1}(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}\,. \tag{4.21}$$

In the first and second relations in (4.19), we have used $\dot{H} \ll H^2$ and $1 \ll |f(\eta) - \xi|$, respectively. The second term is the same as the second one on the r.h.s. of Eq. (2.4), which is given by Eq. (4.11). In this case, the first term is proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2q]}$, the second and fourth terms are proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}$, and the third term is proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2]}$. Hence, the first term is the leading one. The coefficient of the first term cannot be zero because $h_{\mathrm{s}} \neq 0$ and $f_{\mathrm{s}} \neq 0$ and therefore in this case Eq. (4.1) cannot be a solution for the finite-time future singularities.

For case (ii) $[q > 1,\ \sigma < 0]$, the first term evolves as

$$\left(2\dot{H} + 3H^2\right)\left(1 + f(\eta) - \xi\right) \sim 3H^2\left(1 + f(\eta) - \xi\right) \sim 3H^2\left(1 - \xi_{\mathrm{c}}\right) \sim 3h_{\mathrm{s}}^2\left(1 - \xi_{\mathrm{c}}\right)(t_{\mathrm{s}} - t)^{-2q}\,, \tag{4.22}$$

where in the first and second relations, we have used $\dot{H} \ll H^2$ and $f(\eta) \sim 0$, respectively. The second, third, and fourth terms are given by Eqs. (4.11), (4.20), and (4.21), respectively. In this case, the first term is proportional to $(t_{\mathrm{s}} - t)^{-2q}$, the second and fourth terms are

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proportional to $(t_s - t)^{-[(q-1)\sigma + q + 1]}$, and the third term is proportional to $(t_s - t)^{-[(q-1)\sigma + 2]}$. Hence, the first term is the leading one. From Eq. (4.22), we find the condition that the first term vanishes is given by $\xi_c = 1$, which is equivalent to the condition in Eq. (4.14).

For case (iii) $[-1 < q < 0 \,, 0 < q < 1]$, the first term, the former and latter parts of the third one, and the fourth one evolve as

$$\left(2\dot{H} + 3H^2\right)(1 + f(\eta) - \xi) \sim 2\dot{H}(1 + f(\eta) - \xi)$$
$$\sim 2qh_s^2\left(1 + f_s\eta_c^\sigma - \xi_c\right)(t_s - t)^{-(q+1)} \,, \tag{4.23}$$

$$\frac{d^2 f(\eta)}{dt^2} \sim 36h_s^2 f_s\sigma(\sigma - 1)\eta_c^{\sigma-2}(t_s - t)^{-2q} - 6h_s q f_s\sigma\eta_c^{\sigma-1}(t_s - t)^{-(q+1)} \,, \tag{4.24}$$

$$-\frac{d^2\xi}{dt^2} \sim -6h_s q f_s\sigma\eta_c^{\sigma-1}(t_s - t)^{-(q+1)} \,, \tag{4.25}$$

$$2H\frac{d(f(\eta) - \xi)}{dt} \sim -24h_s^2 f_s\sigma\eta_c^{\sigma-1}(t_s - t)^{-2q} \,. \tag{4.26}$$

Here, $d^2 f(\eta)/dt^2$ in Eq. (4.24) and $-d^2\xi/dt^2$ in Eq. (4.25) are the former and latter parts of the third term on the r.h.s. of Eq. (2.5). The second term is given by Eq. (4.16). In the first relation in Eq. (4.23), we have used $\dot{H} \gg H^2$. In this case, the first term, the second part of the former part of the third one given by Eq. (4.24), and the latter part of the third one given by Eq. (4.25) are proportional to $(t_s - t)^{-(q+1)}$, whereas the second term, the first part of the former part of the third one given by Eq. (4.24), and the fourth one are proportional to $(t_s - t)^{-2q}$. The terms proportional to $(t_s - t)^{-(q+1)}$ are the leading ones. From Eqs. (4.23), (4.24) and (4.25), we find the condition that the leading terms vanish is given by $f_s\eta_c^{\sigma-1}(6\sigma - \eta_c) + \xi_c - 1 = 0$, which is the same as the one in Eq. (4.18).

As a consequence, for case (ii) $[q > 1\,, \sigma < 0]$, if the condition $\xi_c = 1$ in Eq. (4.14) is satisfied, the leading term vanishes in both Eqs. (2.4) and (2.5). Furthermore, for case (iii) $[-1 < q < 0\,, 0 < q < 1]$, when the condition $f_s\eta_c^{\sigma-1}(6\sigma - \eta_c) + \xi_c - 1 = 0$ in Eq. (4.18) is met, the leading terms also vanish in both Eqs. (2.4) and (2.5). Thus, the expression of the Hubble parameter H in Eq. (4.1) can be a leading-order solution in terms of $(t_s - t)$ for the gravitational field equations in the flat FLRW space-time. This implies that there can exist the finite-time future singularities in non-local gravity.

C. Analysis for $\eta_c = 0$

If we take $\eta_c = 0$ in Eqs. (4.4) and (4.5), by using Eqs. (4.7) we see that for $q > 1$, ξ is given by Eq. (4.8), whereas for $q < 1$, ξ is described as

$$\xi \sim -\frac{qf_s}{(q-1)\sigma+1}\left(-\frac{6h_s}{q-1}\right)^\sigma (t_s - t)^{-(q-1)\sigma} + \xi_c \quad (-1 < q < 0\,, 0 < q < 1)\,. \qquad (4.27)$$

If the power in terms of $(t_s - t)$ is negative (positive), $-(q-1)\sigma < 0$ (> 0), the first term proportional to $(t_s - t)^{-(q-1)\sigma}$ (the second constant term) is the leading one. Therefore, for $[-1 < q < 0\,, 0 < q < 1\,, \sigma > 0]$, $\xi \sim \xi_c$, while for $[-1 < q < 0\,, 0 < q < 1\,, \sigma < 0]$, $\xi \propto (t_s - t)^{-(q-1)\sigma}$.

Thus, case (iii) is divided into two cases. One is case (iii-a) $[-1 < q < 0\,, 0 < q < 1\,, \sigma > 0]$: $\eta \propto (t_s - t)^{-(q-1)}$ and $\xi \sim \xi_c$. Another is case (iii-b) $[-1 < q < 0\,, 0 < q < 1\,, \sigma < 0]$: $\eta \propto (t_s - t)^{-(q-1)}$ and $\xi \propto (t_s - t)^{-(q-1)\sigma}$.

First, we investigate each term on the r.h.s. of Eq. (2.4). For case (iii-a) $[-1 < q < 0\,, 0 < q < 1\,, \sigma > 0]$, the first, second and third terms evolve as

$$-3H^2\left(1 + f(\eta) - \xi\right) \sim -3H^2 f(\eta) \sim -3h_s^2 f_s\left(-\frac{6h_s}{q-1}\right)^\sigma (t_s - t)^{-[(q-1)\sigma+2q]}\,, \qquad (4.28)$$

$$\frac{1}{2}\dot{\xi}\dot{\eta} \sim -18h_s^2\sigma f_s\left(-\frac{6h_s}{q-1}\right)^{\sigma-1}\frac{q}{(q-1)\sigma+1}(t_s - t)^{-[(q-1)\sigma+q+1]}\,, \qquad (4.29)$$

$$-3H\left(\frac{df(\eta)}{d\eta}\dot{\eta} - \dot{\xi}\right) \sim 18h_s^2\sigma f_s\left(-\frac{6h_s}{q-1}\right)^{\sigma-1}\frac{(q-1)\sigma+q+1}{(q-1)\sigma+1}(t_s - t)^{-[(q-1)\sigma+q+1]}\,. \qquad (4.30)$$

In the first relation in (4.28), we have used $|1 - \xi_c| \ll |f(\eta)|$. In this case, the first term is proportional to $(t_s - t)^{-[(q-1)\sigma+2q]}$, whereas the second and third terms are proportional to $(t_s - t)^{-[(q-1)\sigma+q+1]}$. Hence, the second and third terms are the leading ones and the coefficient of these leading terms is given by $18h_s^2\sigma f_s\left[-6h_s/(q-1)\right]^{\sigma-1}$. This cannot be zero because $h_s \neq 0$, $f_s \neq 0$ and $\sigma \neq 0$ and hence the leading terms cannot vanish. This implies that in this case, $H \sim h_s/(t_s - t)^q$ in Eq. (4.1) cannot be a solution describing the finite-time future singularities.

For case (iii-b) $[-1 < q < 0\,, 0 < q < 1\,, \sigma < 0]$, the first term evolves as

$$-3H^2\left(1 + f(\eta) - \xi\right) \sim -3H^2\left(f(\eta) - \xi\right)$$
$$\sim -3h_s^2 f_s\left(-\frac{6h_s}{q-1}\right)^\sigma\frac{(q-1)\sigma+q+1}{(q-1)\sigma+1}(t_s - t)^{-[(q-1)\sigma+2q]}\,, \qquad (4.31)$$

where in the first relation, we have used $1 \ll |f(\eta) - \xi|$. The second and third terms are given by Eqs. (4.29) and (4.30), respectively. Since the first term is proportional to

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$(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2q]}$, which is the same as in Eq. (4.28), the second and third terms are the leading ones. Thus, the consequence is equivalent to that in case (iii-a).

Next, we investigate the r.h.s. of Eq. (2.5). For case (iii-a) $[-1 < q < 0\,,\, 0 < q < 1\,,$ $\sigma > 0]$, the first, third and fourth terms evolve as

$$
\left(2\dot{H} + 3H^2\right)(1 + f(\eta) - \xi) \sim 2\dot{H}(1 + f(\eta) - \xi) \sim -2\dot{H}f(\eta)
$$
$$
\sim 2h_{\mathrm{s}}f_{\mathrm{s}}\left(-\frac{6h_{\mathrm{s}}}{q-1}\right)^{\sigma}(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}\,, \tag{4.32}
$$

$$
\frac{d^2\left(f(\eta) - \xi\right)}{dt^2} \sim 36h_{\mathrm{s}}^2 f_{\mathrm{s}}\sigma\left(-\frac{6h_{\mathrm{s}}}{q-1}\right)^{\sigma-2}\frac{(q-1)\sigma+q+1}{q-1}(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2]}\,, \tag{4.33}
$$

$$
2H\frac{d\left(f(\eta) - \xi\right)}{dt} \sim -12h_{\mathrm{s}}^2 f_{\mathrm{s}}\sigma\left(-\frac{6h_{\mathrm{s}}}{q-1}\right)^{\sigma-1}\frac{(q-1)\sigma+q+1}{(q-1)\sigma+1}(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}\,. \tag{4.34}
$$

In the first and second relations in (4.32), we have used $\dot{H} \gg H^2$ and $|1 - \xi_{\mathrm{c}}| \ll |f(\eta)|$, respectively. The second term is the same as the second one on the r.h.s. of Eq. (2.4), which is given by Eq. (4.29). In this case, the first, second and fourth terms are proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}$, whereas the third term $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+2]}$. Hence, the third term is the leading one. From Eq. (4.33), we find the condition that the third term vanishes is given by

$$
\sigma = -\frac{q+1}{q-1}\,. \tag{4.35}
$$

For case (iii-b) $[-1 < q < 0\,,\, 0 < q < 1,\, \sigma < 0]$, the first term evolves as

$$
\left(2\dot{H} + 3H^2\right)(1 + f(\eta) - \xi) \sim 2\dot{H}(1 + f(\eta) - \xi) \sim 2\dot{H}(f(\eta) - \xi)
$$
$$
\sim 2qh_{\mathrm{s}}f_{\mathrm{s}}\left(-\frac{6h_{\mathrm{s}}}{q-1}\right)^{\sigma}\frac{(q-1)\sigma+q+1}{(q-1)\sigma+1}(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}\,, \tag{4.36}
$$

where in the first and second relation, we have used $\dot{H} \gg H^2$ and $1 \ll |f(\eta) - \xi|$, respectively. The second, third, and fourth terms are given by Eqs. (4.29), (4.33) and (4.34), respectively. Since the first term is proportional to $(t_{\mathrm{s}} - t)^{-[(q-1)\sigma+q+1]}$, which is the same as in Eq. (4.32), the third term is the leading one. Thus, the consequence is equivalent to that in case (iii-a).

As a result, if we take $\eta_{\mathrm{c}} = 0$, we cannot have the case in which the leading term vanishes in both Eqs. (2.4) and (2.5).

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D.   Relations between the model parameters and the property of the finite-time future singularities

The parameters $f_{\rm s}$ and σ in the form of $f(\eta)$ in Eq. (4.6) characterize the theory of non-local gravity described by the action in Eq. (2.1). On the other hand, the parameters $h_{\rm s}$, $t_{\rm s}$ and q in the expression of the Hubble parameter H in Eq. (4.1) specify the property of the finite-time future singularity. Moreover, the integration constants $\eta_{\rm c}$ in Eqs. (4.4) and (4.5) and $\xi_{\rm c}$ in Eqs. (4.8) and (4.9) determine a leading-order solution in terms of $(t_{\rm s} - t)$ for the gravitational field equations in the flat FLRW space-time.

From Eq. (4.2), we see that when $t \to t_{\rm s}$, for $q > 1$, $a \to \infty$, whereas for $-1 < q < 0$ and $0 < q < 1$, $a \to a_{\rm s}$. Moreover, it follows from Eqs. (4.1) and (2.44) that for $q > 0$, $H \to \infty$ and therefore $\rho_{\rm eff} = 3H^2/\kappa^2 \to \infty$, whereas for $-1 < q < 0$, H asymptotically becomes finite and also $\rho_{\rm eff}$ asymptotically approaches a finite constant value $\rho_{\rm s}$. On the other hand, from $\dot{H} \sim qh_{\rm s} (t_{\rm s} - t)^{-(q+1)}$ and Eq. (2.45) we find that for $q > -1$, $\dot{H} \to \infty$ and hence $P_{\rm eff} = - \left(2\dot{H} + 3H^2\right)/\kappa^2 \to \infty$.

It is known that the finite-time future singularities[1] can be classified in the following manner [33]:

- Type I ("Big Rip" [34]): In the limit $t \to t_{\rm s}$, $a \to \infty$, $\rho_{\rm eff} \to \infty$ and $|P_{\rm eff}| \to \infty$. The case in which $\rho_{\rm eff}$ and $P_{\rm eff}$ becomes finite values at $t = t_{\rm s}$ [35] is also included.

- Type II ("sudden" [15, 36]): In the limit $t \to t_{\rm s}$, $a \to a_{\rm s}$, $\rho_{\rm eff} \to \rho_{\rm s}$ and $|P_{\rm eff}| \to \infty$.

- Type III: In the limit $t \to t_{\rm s}$, $a \to a_{\rm s}$, $\rho_{\rm eff} \to \infty$ and $|P_{\rm eff}| \to \infty$.

- Type IV: In the limit $t \to t_{\rm s}$, $a \to a_{\rm s}$, $\rho_{\rm eff} \to 0$, $|P_{\rm eff}| \to 0$, and higher derivatives of H diverge. The case in which $\rho_{\rm eff}$ and/or $|P_{\rm eff}|$ asymptotically approach finite values is also included.

As a consequence, the finite-time future singularities described by the expression of H in Eq. (4.1) in non-local gravity have the following properties: For $q > 1$, the Type I ("Big Rip") singularity, for $0 < q < 1$, the Type III singularity, and for $-1 < q < 0$, the Type II ("sudden") singularity.

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[1]   For the first observation of the finite-time future singularities in $F(R)$ gravity, see [31]. In Ref. [32], the finite-time future singularities in various modified gravity theories have been examined.

TABLE I: Range and conditions for the value of parameters of $f(\eta)$, H, and η_c and ξ_c in order that the finite-time future singularities can exist.

Case	$f(\eta) = f_s \eta^\sigma$	$H \sim \frac{h_s}{(t_s - t)^q}$	η_c, ξ_c
	$f_s \neq 0$	$h_s > 0$	$\eta_c \neq 0$
	$\sigma \neq 0$	$q > -1$, $q \neq 0$	
(ii)	$\sigma < 0$	$q > 1$ [Type I ("Big Rip") singularity]	$\xi_c = 1$
(iii)	$f_s \eta_c^{\sigma-1} (6\sigma - \eta_c) + \xi_c - 1 = 0$	$0 < q < 1$ [Type III singularity]	
		$-1 < q < 0$ [Type II ("sudden") singularity]	

The results obtained in Secs. IV A and IV B are summarized in Table I. In Table I, we show the range and conditions for the value of parameters of $f(\eta)$, H, and η_c and ξ_c in order that the finite-time future singularities can exist. If $\eta_c \neq 0$ and $\xi_c = 1$, in a model with $\sigma < 0$, there can exist the finite-time future singularities with the property of the Type I ("Big Rip") singularity for $q > 1$. If $\eta_c \neq 0$, in a model with satisfying the condition $f_s \eta_c^{\sigma-1} (6\sigma - \eta_c) + \xi_c - 1 = 0$, there can exist the finite-time future singularities with the property of the Type III singularity for $0 < q < 1$ and that of the Type II ("sudden") singularity for $-1 < q < 0$. We remark that for the special case of $\eta_c = 0$, the finite-time future singularities described by H in Eq. (4.1) cannot occur.

V. EFFECTIVE EQUATION OF STATE FOR THE UNIVERSE AND PHANTOM-DIVIDE CROSSING

A. Cosmological evolution of the effective equation of state for the universe

We study the effective equation of state for the universe when the finite-time future singularities occur in non-local gravity. We examine the asymptotic behavior of w_{eff} given by Eq. (2.54) in the limit $t \to t_s$ by using the results obtained in Sec. IV B. First, we study case (ii) [$q > 1$, $\sigma < 0$] [Type I ("Big Rip") singularity]. The first term in the denominator of Eq. (2.54) is given by $-3H^2 (f(\eta) - \xi) \sim 3h_s^2 (t_s - t)^{-2q}$, where we have used $f(\eta) \sim 0$ and Eq. (4.14). Moreover, the first term in the numerator of Eq. (2.54) is

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given by $\left(2\dot{H} + 3H^2\right)(f(\eta) - \xi) \sim -3h_{\mathrm{s}}^2 (t_{\mathrm{s}} - t)^{-2q}$, where we have used $\dot{H} \ll H^2$, $f(\eta) \sim 0$ and $\xi_{\mathrm{c}} = 1$. We note that these first terms proportional to $(t_{\mathrm{s}} - t)^{-2q}$ are the leading ones in the denominator and numerator of Eq. (2.54) and therefore Λ, $\kappa^2 \rho_{\mathrm{m}}$ and $\kappa^2 P_{\mathrm{m}}$ can be neglected. By substituting these relations, Eqs. (4.11), (4.12), (4.20) and (4.21) into Eq. (2.54) and taking the leading term in terms of $(t_{\mathrm{s}} - t)$, we find

$$w_{\mathrm{eff}} \sim -1 + I(t) \sim -1\,, \tag{5.1}$$

where

$$I(t) = -8\sigma f_{\mathrm{s}} \left(-\frac{4h_{\mathrm{s}}}{q-1}\right)^{\sigma-1} (t_{\mathrm{s}} - t)^{(q-1)(1-\sigma)}\,. \tag{5.2}$$

Here, $I(t)$ corresponds to the deviation of w_{eff} from -1. In deriving the second relation in (5.1), we have used the fact that $(t_{\mathrm{s}} - t)^{(q-1)(1-\sigma)} \sim 0$ because the power in terms of $(t_{\mathrm{s}} - t)$ is positive $((q-1)(1-\sigma) > 0)$. Since $h_{\mathrm{s}} > 0$, $q > 1$ and $\sigma < 0$, from (5.1) and Eq. (5.2) we see that if $(-)^{\sigma-1} f_{\mathrm{s}} > 0$, $I(t)$ evolves from $I(t) > 0$ to $I(t) = 0$, and hence w_{eff} evolves from $w_{\mathrm{eff}} > -1$: the non-phantom (quintessence) phase and asymptotically approaches $w_{\mathrm{eff}} = -1$ corresponding to the phase of the cosmological constant, whereas if $(-)^{\sigma-1} f_{\mathrm{s}} < 0$, $I(t)$ evolves from $I(t) < 0$ to $I(t) = 0$, and therefore w_{eff} evolves from $w_{\mathrm{eff}} < -1$: the phantom phase and asymptotically approaches $w_{\mathrm{eff}} = -1$.

Next, we explore case (iii) $[-1 < q < 0\,, 0 < q < 1]$. The first term in the denominator of Eq. (2.54) is given by $-3H^2(f(\eta) - \xi) \sim -3h_{\mathrm{s}}^2 (f_{\mathrm{s}}\eta_{\mathrm{c}}^\sigma - \xi_{\mathrm{c}})(t_{\mathrm{s}} - t)^{-2q}$. Moreover, the first term in the numerator of Eq. (2.54) is given by $\left(2\dot{H} + 3H^2\right)(f(\eta) - \xi) \sim 2qh_{\mathrm{s}}^2 (1 + f_{\mathrm{s}}\eta_{\mathrm{c}}^\sigma - \xi_{\mathrm{c}})(t_{\mathrm{s}} - t)^{-(q+1)}$, where we have used $\dot{H} \gg H^2$. By substituting these relations, Eqs. (4.16), (4.17), (4.24), (4.25) and (4.26) into Eq. (2.54) and using the condition in Eq. (4.18), we obtain the expression of Eq. (2.54) and find that in the denominator of Eq. (2.54) there is a term proportional to $(t_{\mathrm{s}} - t)^{-2q}$ and in the numerator there are a term proportional to $(t_{\mathrm{s}} - t)^{-2q}$ and that proportional to $(t_{\mathrm{s}} - t)^{-(q+1)}$. If the power of the terms proportional to $(t_{\mathrm{s}} - t)^{-2q}$ is negative, i.e., for $0 < q < 1$ [Type III singularity], the power of the term proportional to $(t_{\mathrm{s}} - t)^{-(q+1)}$ is also negative, and therefore in the limit $t \to t_{\mathrm{s}}$ the terms proportional to $(t_{\mathrm{s}} - t)^{-2q}$ and that proportional to $(t_{\mathrm{s}} - t)^{-(q+1)}$ become dominant over $(\Lambda + \kappa^2 \rho_{\mathrm{m}})$ and $(-\Lambda + \kappa^2 P_{\mathrm{m}})$. Hence, Λ, $\kappa^2 \rho_{\mathrm{m}}$ and $\kappa^2 P_{\mathrm{m}}$ can be neglected. As a consequence, Eq. (2.54) is described as

$$w_{\mathrm{eff}} \sim -1 + I(t) \sim -\frac{2q}{3h_{\mathrm{s}}}(t_{\mathrm{s}} - t)^{q-1}\,, \tag{5.3}$$

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where

$$I(t) = I_0 - \frac{2q}{3h_{\mathrm{s}}} (t_{\mathrm{s}} - t)^{q-1} , \qquad (5.4)$$

$$I_0 = 1 + 2f_{\mathrm{s}}\sigma\eta_{\mathrm{c}}^{\sigma-2} \left[6\left(\sigma - 1\right) - 7\eta_{\mathrm{c}} \right] . \qquad (5.5)$$

Here, I_0 is a constant part of $I(t)$. In deriving the second relation in (5.3), we have used the fact that in the limit $t \to t_{\mathrm{s}}$, the second term in Eq. (5.4) becomes dominant over the first one because the power in terms of $(t_{\mathrm{s}} - t)$ is negative $(q - 1 < 0)$. If $I_0 > 0$, a crossing of the phantom divide from the non-phantom phase to the phantom one can occur because the sign of the second term in Eq. (5.4) is negative and the absolute value of the amplitude becomes very large, while if $I_0 < 0$, $I(t)$ always evolves in the phantom phase ($w_{\mathrm{eff}} < -1$). Thus, for $0 < q < 1$ the final stage is the phantom phase and it is eternal. It seems that the recent cosmological observational data imply the crossing of the phantom divide from the non-phantom phase to the phantom one in the near past [37]. The (past or) future crossing of the phantom divide in $F(R)$ gravity have also been investigated in Ref. [38].

On the other hand, if $-1 < q < 0$ [Type II ("sudden") singularity], in the denominator of Eq. (2.54) the term proportional to $(t_{\mathrm{s}} - t)^{-2q}$ asymptotically vanishes and $(\Lambda + \kappa^2\rho_{\mathrm{m}})$ become dominant, whereas in the numerator the term proportional to $(t_{\mathrm{s}} - t)^{-(q+1)}$ again becomes dominant over $(-\Lambda + \kappa^2 P_{\mathrm{m}})$, and thus Λ and $\kappa^2 P_{\mathrm{m}}$ can be neglected. As a result, we acquire

$$w_{\mathrm{eff}} \sim -2h_{\mathrm{s}}q\,(t_{\mathrm{s}} - t)^{-(q+1)} / \left(\Lambda + \kappa^2\rho_{\mathrm{m}}\right) . \qquad (5.6)$$

The absolute value of the amplitude of the term $-2h_{\mathrm{s}}q\,(t_{\mathrm{s}} - t)^{-(q+1)}$ becomes very large. If we consider $\Lambda > 0$, since $h_{\mathrm{s}} > 0$ and $-1 < q < 0$, we have $w_{\mathrm{eff}} > 0$.

In summary, for $q > 1$ [Type I ("Big Rip") singularity], w_{eff} evolves from the non-phantom phase or the phantom one and asymptotically approaches $w_{\mathrm{eff}} = -1$. For $0 < q < 1$ [Type III singularity], w_{eff} evolves from the non-phantom to the phantom one with realizing a crossing of the phantom divide or evolves in the phantom phase. The final stage is the eternal phantom phase. For $-1 < q < 0$ [Type II ("sudden") singularity], $w_{\mathrm{eff}} > 0$ at the final stage.

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B. Estimation of the current value of the effective equation of state parameter for non-local gravity

In Ref. [3], the limit on a constant equation of state for dark energy in a flat universe has been estimated as $w_{\mathrm{DE}} = -1.10 \pm 0.14\,(68\%\,\mathrm{CL})$ by combining the data of Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations [3] with the latest distance measurements from the BAO in the distribution of galaxies and the Hubble constant measurement. Moreover, for a time-dependent equation of state for dark energy, by using a linear form $w_{\mathrm{DE}}(a) = w_{\mathrm{DE}\,0} + w_{\mathrm{DE}\,a}\,(1-a)$ [39] with $w_{\mathrm{DE}\,0}$ and $w_{\mathrm{DE}\,a}$ being the current value of w_{DE} and its derivative, constraints on $w_{\mathrm{DE}\,0}$ and $w_{\mathrm{DE}\,a}$ have been found as $w_{\mathrm{DE}\,0} = -0.93 \pm 0.13$ and $w_{\mathrm{DE}\,a} = -0.41^{+0.72}_{-0.71}\,(68\%\,\mathrm{CL})$, from the combination of the WMAP data with the BAO data, the Hubble constant measurement and the high-redshift SNe Ia data.

We estimate the present value of w_{eff}. Here, we regard w_{eff} as being approximately equal to w_{DE} at the present time ($w_{\mathrm{eff}} \approx w_{\mathrm{DE}}$) because the energy density of dark energy is dominant over that of non-relativistic matter at the present time. For case (ii) $[q > 1, \sigma < 0]$, we take $\sigma = -1$, $q = 2$, $h_{\mathrm{s}} = 1\,[\mathrm{GeV}]^{-1}$ and $t_{\mathrm{s}} = 2t_{\mathrm{p}}$, where t_{p} is the present time. The current value of the Hubble parameter is given by $H_{\mathrm{p}} = 2.1h \times 10^{-42}\mathrm{GeV}$ [40] with $h = 0.7$ [3, 41]. We note that in Eq. (4.1), h_{s} has the dimension of $[\mathrm{Mass}]^{q-1}$ and t_{s} and t have the dimension of $[\mathrm{Mass}]^{-1}$. In this case, by using the first approximate relation in (5.1) and (5.2), we find that if $f_{\mathrm{s}} = -3.0 \times 10^{-43}$, $w_{\mathrm{eff}} = -1.10$, and if $f_{\mathrm{s}} = -2.1 \times 10^{-43}$, $w_{\mathrm{eff}} = -0.93$. For $0 < q < 1$, we take $\sigma = 1$, $q = 1/2$, $h_{\mathrm{s}} = 1\,[\mathrm{GeV}]^{1/2}$, $\eta_{\mathrm{c}} = 1$ and $t_{\mathrm{s}} = 2t_{\mathrm{p}}$. In this case, by using the first approximate relation in (5.3) and Eqs. (5.4) and (5.5), we see that if $f_{\mathrm{s}} = 7.9 \times 10^{-2}$, $w_{\mathrm{eff}} = -1.10$, and if $f_{\mathrm{s}} = 6.6 \times 10^{-2}$, $w_{\mathrm{eff}} = -0.93$. For $-1 < q < 0$, from Eq. (5.6) we have $w_{\mathrm{eff}} > 0$. Thus, in our models w_{eff} can have the present observed value of w_{DE} estimated in Ref. [3].

We remark that in this subsection, as a demonstration we have examined whether our model can have the observed value of w_{DE} at the present time. Therefore, at the current stage it seems it might not be stated that this model has an improvement to other dark energy models. However, an advantage of non-local gravity is that there could exist a possible scenario to screen the cosmological constant, although there still remains the problem of the existence of the ghost. This may be a clue in order to solve the smallness problem of the cosmological constant.

C. Cosmological consequences of adding an R^2 term

We investigate the cosmological consequences of adding an R^2 term. It is known that in $F(R)$ gravity, the addition of an R^2 term could cure the finite-time future singularities [[31], in which the introduction of an R^2 term for curing the Big Rip singularity was first proposed, the first reference in Ref. [8], [32, 42]]. We explore whether the addition of an R^2 term removes the finite-time future singularities in non-local gravity. We examine the case in which $uR^2/(2\kappa^2)$ is added to the action in Eq. (2.1), i.e., the action is described by Eq. (2.48). If the Hubble parameter is expressed as Eqs. (4.1), in the limit $t \to t_{\rm s}$, Θ in Eq. (2.52) and Ξ in Eq. (2.53) are approximately given by

$$\Theta \sim 18u \left[-6h_{\rm s}^2 q\,(t_{\rm s}-t)^{-(3q+1)} + h_{\rm s}^2 q^2\,(t_{\rm s}-t)^{-2(q+1)} - 2h_{\rm s}^2 q\,(q+1)\,(t_{\rm s}-t)^{-2(q+1)} \right] , \quad (5.7)$$

$$\Xi \sim 6u \left[9h_{\rm s}^2 q^2\,(t_{\rm s}-t)^{-2(q+1)} + 18h_{\rm s}^3 q\,(t_{\rm s}-t)^{-(3q+1)} \right.$$
$$\left. + 2h_{\rm s} q\,(q+1)\,(q+2)\,(t_{\rm s}-t)^{-(q+3)} + 12h_{\rm s}^2 q\,(q+1)\,(t_{\rm s}-t)^{-2(q+1)} \right] . \quad (5.8)$$

We consider the r.h.s. of Eq. (2.4) with Θ in Eq. (2.52). For case (ii) $[q > 1, \sigma < 0]$, the first term of Eq. (5.7), $-108uh_{\rm s}^2 q\,(t_{\rm s}-t)^{-(3q+1)}$, becomes the leading one. Since $u \neq 0$, $h_{\rm s} \neq 0$ and $q \neq 0$, this leading term does not vanish. This means that the additional R^2 term can remove the finite-time future singularity. Furthermore, for case (iii) $[-1 < q < 0, 0 < q < 1]$, the second and third terms of Eq. (5.7), whose summation is given by $-18uh_{\rm s}^2 q\,(q+2)\,(t_{\rm s}-t)^{-2(q+1)}$, becomes the leading ones. Since $u \neq 0$, $h_{\rm s} \neq 0$, $q \neq 0$ and $q \neq -2$, these leading terms do not vanish. This means that the additional R^2 term can cure the finite-time future singularity. The above qualitative analysis shows that adding an R^2 term makes the solutions of Secs. II and III more complicated, admitting another de Sitter space. Such a de Sitter solution may be an origin of the inflationary stage. We remark that a natural non-local gravity term which may probably cure the finite-time future singularities is $R^\upsilon \Box^{-1} R$, where υ is a constant and could be close to 2. As this term is similar to a pure $R^2 \Box^{-1}$ operator acting on R, it cancels a number of derivative terms.

Finally, it is interesting to mention that in the context of cosmology, the addition of an R^2 term might also be meaningful. In Ref. [11], it has been suggested that in the framework of non-local gravity combined with an $F(R)$ term, inflation in the early universe as well as the cosmic acceleration in the late time could be realized. The additional R^2 term leads to inflation and the late-time cosmic acceleration occurs due to the term of non-local gravity

$Rf(\Box^{-1}R)$ in the action in Eq. (2.1). The unified scenario of inflation and dark energy in such a theory occurs qualitatively in the same way as in the models of $F(R)$ gravity [8].

VI. CONCLUSION

In this paper, we have studied de Sitter solutions not only in non-local gravity but also in non-local gravity with Lagrange constraint multiplier. We have explored a condition to avoid a ghost and presented a screening scenario for a cosmological constant in de Sitter solutions. In addition, we have explicitly shown that three types of the finite-time future singularities can occur in non-local gravity and examined their properties. Furthermore, we have investigated the behavior of the effective equation of state for the universe when the finite-time future singularities occur and shown that the late-time accelerating universe may be effectively the quintessence, cosmological constant or phantom-like phases. We have also demonstrated that there is a case with realizing a crossing of the phantom divide from the non-phantom (quintessence) phase to the phantom one in the limit of the appearance of a finite-time future singularity. The estimation of the current value of the effective equation of state parameter for the universe which could be phantomic one around -1 shows that its observed value could be easily realized by the appropriate choice of non-local gravity parameters. Moreover, we have considered the cosmological consequences of adding an R^2 term and demonstrated that the addition of an R^2 term can remove the finite-time future singularities in non-local gravity. It has also been suggested that the addition of an R^2 term in the framework of non-local gravity might realize unification of inflation in the early universe with the cosmic acceleration in the late time. Nevertheless, it is interesting to understand if there exists some non-local gravity action which cures the finite-time future singularities in the similar way.

Appendix A: Derivation of the leading term of η in terms of $(t_{\rm s} - t)$

In this appendix, we explain the derivation of Eqs. (4.4) and (4.5) from Eq. (4.3) in Sec. IV A.

For $q > 1$, the integration part in terms of $\bar{t}$ of Eq. (4.3) is given by

$$- \int^{\bar{t}} R a^3 d\bar{t} = \int^{\tilde{t}} R a^3 d\tilde{t} \sim 12 h_{\rm s}^2 a_{\rm s}^3 J_1 \,, \tag{A1}$$

where $\tilde{t} \equiv (t_{\rm s} - t)$ and J_1 is defined as

$$J_1 \equiv \int \tilde{t}^{-2q} \exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] d\tilde{t} \tag{A2}$$

$$= \frac{1}{(-3h_{\rm s})} \int \tilde{t}^{-2q} \tilde{t}^q \frac{d}{d\tilde{t}}\left\{\exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right]\right\} d\tilde{t} \tag{A3}$$

$$= \frac{1}{(-3h_{\rm s})} \left\{\tilde{t}^{-q} \exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] - (-q) J_2\right\} \,, \tag{A4}$$

with

$$J_2 \equiv \int \tilde{t}^{-(q+1)} \exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] d\tilde{t} \tag{A5}$$

$$= \frac{1}{(-3h_{\rm s})} \int \tilde{t}^{-(q+1)} \tilde{t}^q \frac{d}{d\tilde{t}}\left\{\exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right]\right\} d\tilde{t} \tag{A6}$$

$$= \frac{1}{(-3h_{\rm s})} \left\{\tilde{t}^{-1} \exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] - (-1) \int \tilde{t}^{-2} \exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] d\tilde{t}\right\} \,. \tag{A7}$$

In the second relation in (A1), we have used Eq. (4.1) and the fact that $\dot{H} \ll H^2$ and therefore $R \sim 12H^2$. Moreover, in the equalities in Eqs. (A4) and (A7), we have executed the partial integration. The leading term in terms of $\tilde{t} = (t_{\rm s} - t)$ in J_1 is the first term in

Eqs. (A4). Using Eqs. (4.2), (A1) and (A4), we find

$$-\int^{\bar{t}} Ra^3 d\bar{t} = \int^{\tilde{t}} Ra^3 d\tilde{t} \sim -4h_{\rm s}\tilde{t}^{-q}a^3 \,. \tag{A8}$$

By combining Eq. (4.3) and (A8) and executing the integration with respect to $\tilde{t}$, we obtain Eq. (4.4).

On the other hand, for $-1 < q < 0$ and $0 < q < 1$, the integration part in terms of $\bar{t}$ of Eq. (4.3) is given by

$$-\int^{\bar{t}} Ra^3 d\bar{t} = \int^{\tilde{t}} Ra^3 d\tilde{t} \sim 6qh_{\rm s}a_{\rm s}^3 \int \tilde{t}^{-(q+1)} \exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] d\tilde{t} \,, \tag{A9}$$

where

$$\exp\left[\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] = 1 + \frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)} + \frac{1}{2}\left(\frac{3h_{\rm s}}{q-1}\right)^2 \tilde{t}^{-2(q-1)} + \dots \,. \tag{A10}$$

In the second relation in Eq. (A9), we have used Eq. (4.1) and the fact that $\dot{H} \gg H^2$ and hence $R \sim 6\dot{H}$. Since we consider the case that $-1 < q < 0$ and $0 < q < 1$, in the limit of $t \to t_{\rm s}$, i.e., $\tilde{t} \to 0$, from Eq. (A10) we see that the first term on the r.h.s. of Eq. (A10) is the leading one. Hence, we have

$$-\int^{t'} Ra^3 dt' = \int^{\tilde{t}} Ra^3 d\tilde{t} \sim -6h_{\rm s}a_{\rm s}^3\tilde{t}^{-q} \,. \tag{A11}$$

Substituting Eq. (A11) into Eq. (4.3) and using Eq. (4.2), we find

$$\eta = 6h_{\rm s}\int \tilde{t}^{-q} \exp\left[-\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] d\tilde{t} \,, \tag{A12}$$

where

$$\exp\left[-\frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)}\right] = 1 - \frac{3h_{\rm s}}{q-1}\tilde{t}^{-(q-1)} + \frac{1}{2}\left(-\frac{3h_{\rm s}}{q-1}\right)^2 \tilde{t}^{-2(q-1)} + \dots \,. \tag{A13}$$

Similarly to Eq. (A10), the first term on the r.h.s. of Eq. (A13) is the leading one. Thus, by executing the integration with respect to $\tilde{t}$ in Eq. (A12) and taking the leading term in terms of $\tilde{t} = (t_{\rm s} - t)$, we acquire Eq. (4.5).

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